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Exact statistical properties of the Burgers equation

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The one-dimensional Burgers equation in the inviscid limit with white noise initial condition is revisited. The one- and two-point distributions of the Burgers field as well as the related distributions of shocks are obtained in closed analytical forms. In particular, the large distance behaviour of spatial correlations of the field is determined. Since higher-order distributions factorize in terms of the one- and two-point functions, our analysis provides an explicit and complete statistical description of this problem.

1. Introduction

The Burgers equation for the velocity field $u(x,t)$,

$$\frac{\partial}{\partial t}u(x,t)+u(x,t)\frac{\partial}{\partial x}u(x,t)=\nu\frac{\partial^2}{\partial x^2}u(x,t), \qquad (1)$$

has raised much interest because of its multiple connections to a variety of physical and mathematical problems. Background and references can be found for instance in Gurbatov, Malakhov & Saichev (1991) and Woyczyński (1998). We cannot do justice here to the very large literature on the subject and all the physical applications, but just recall some aspects. On one hand, equation (1) is a version of the one-dimensional Navier–Stokes equation without pressure and external forces. Although it can be solved with the Hopf–Cole transformation (to be recalled below), the determination of the statistics of the velocity field and of its large time asymptotics leads to non-trivial problems when the initial data are chosen randomly. It provides an oversimplified, but analytically tractable model of decaying turbulence (Burgers turbulence) which has been much studied in the last decade. On the other hand, the Burgers equation (1) is relevant to the propagation of nonlinear acoustic waves in non-dispersive media. In this interpretation, the role of time and space are inverted; initial velocity field in the case of turbulence becomes a time-dependent source term at a point in space. When the source is random, the statistics of the propagating signal away from the source are of interest (see in particular chap. 13 in Hamilton & Blastock 1998). Let us also mention that the two- and three-dimensional Burgers equations occur in connection with the study of surface growth via the deposition process (it is called the Kardar, Parisi and Zhang (KPZ) equation (Kardar, Parisi & Zhang 1986) in this context and $u(x,t)=\nabla h(x,t)$ with h the profile height of the surface), as well as being a model for investigating the large-scale structure of the universe (the Zeldovich adhesion approximation (Shandarin & Zeldovich 1989)).

In this work, we come back to the original Burgers problem (Burgers 1974) which concerned the statistics of the velocity field $u(x,t)$ and of shock-waves in the inviscid limit $\nu \to 0$ when the distribution of the initial velocity field $u(x,0)$ is a δ-correlated Gaussian (white noise). The choice of white noise, although very singular, is natural in the sense that it corresponds to the assumption of maximal randomness and absence of correlations in the velocity field at a given time (or to a source with flat power spectrum in the acoustic case), the problem being then to understand the nature of the correlations induced by the nonlinearity of the Burgers equation in the course of the time (or away from the source). In Burgers (1974), a considerable amount of work was carried out to calculate various moments of these distributions, but the distribution of the field $p_1(x,u,t) = \mathrm{Prob}\{u(x,t) = u\}$ itself was not obtained in closed form owing to the complexity of the analysis. The question has been addressed again by Tatsumi & Kida (1972) and Kida (1979). In Tatsumi & Kida (1972), kinetic equations for the dynamics of shocks are used to derive scaling properties, and in the second part of Kida (1979), the result of numerical simulations for the distribution of the strength and the velocity of shocks are presented. Recently, Avellaneda and E (1995) and Avellaneda (1995) have derived rigorous upper and lower bounds of the cubic type $\exp(-C|u|^3 t)$ for the tails of the distributions. Such cubic bounds have also been obtained in Martin & Piasecki (1994) for the distribution of mass in the closely related problem of ballistic aggregation. In Ryan (1998*a*), $p_1(u,t)$ is expressed in terms of a certain function satisfying partial differential equations, and more refined bounds are given for the large field asymptotics. For a short review, see Bertoin (2000).

In this paper we provide closed analytical forms of the statistical distributions for the field and the associated distribution of shock-waves. One of our main contributions is a simple formula expressing the shock strength distribution $\rho_1(\mu,t)$ in terms of Airy functions $\mathrm{Ai}(w)$ (equations (75), (76) and (55))

$$\rho_1(\mu,t) = t^{-4/3}\rho_1(\mu t^{-2/3}), \quad \rho_1(\mu) = 2a^3\mu\mathscr{I}(\mu)\mathscr{H}(\mu)$$

with

$$\mathscr{I}(\mu) = \sum_{k\geqslant 1} \mathrm{e}^{-a\omega_k\mu}, \quad \mathscr{H}(\mu) = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}w\, \frac{\mathrm{e}^{-a\mu w}}{\mathrm{Ai}^2(w)} \tag{2}$$

where $-\omega_k$ ($\omega_k > 0$) are the zeros of the Airy function, and a is related to the intensity of the initial white noise. The corresponding asymptotic behaviour for large μ reads (equation (78))

$$\rho_1(\mu) = 2\sqrt{\pi}a^{9/2}\mu^{5/2}\exp\left(-\frac{a^3\mu^3}{12} - \omega_1 a\mu\right), \quad \mu \to \infty. \tag{3}$$

This non-Gaussian behaviour is of course compatible with the bounds found in the above-mentioned works and is illustrated in figure 4. Similar formulae are obtained for the distribution $p_1(u,t)$ of the Burgers field itself and a detailed study of the clustering behaviour of the two-point distribution is presented (§ 5). At any positive time, the initially uncorrelated white-noise field acquires correlations over the whole space, but these correlations are strongly suppressed at large distance by a factor $\exp(-a^3x^3/12t^2)$, $x \to \infty$. Moreover it turns out that the higher-order distributions obey a factorization property in terms of the one- and two-point functions (§ 6). Hence, our results give in fact a complete solution to the Burgers problem with initial white noise distributed data.

Our analysis closely follows the spirit and the methods of Burgers (1974). In § 2, we recall the Hopf–Cole transformation together with well-known facts about the Burgers equation in the inviscid limit, with the purpose of introducing the notation and the definition of the one- and two-point distribution functions. In § 3, using the notion of first hitting time, these distributions are expressed in terms of the basic propagator for Brownian motion constrained by parabolic barriers. This propagator can be explicitly constructed by solving the Airy eigenvalue problem. It appears then that all statistical properties of the Burgers problem are embodied in the knowledge of three functions called here I, J and H. The functions I and J are calculated in § 4 and the one-point distributions of fields and shocks are discussed. These results have already been announced in Frachebourg (1999) in the equivalent language of ballistic aggregation. Section 5 is devoted to the study of the large distance behaviour of the two-point distribution (the function H). Since the analysis is somewhat heavy, technical parts have been relegated to appendices. Finally, the factorization of higher-order distributions is demonstrated in § 6.

One key new ingredient in comparison to Burgers' analysis appears in § 4 in the calculation of the function J (the function I appears in Burgers (1974)). The use of contour integral representation of this function leads to considerable simplifications when a number of remarkable identities between Airy functions are introduced (see e.g. equation (65)). Also the study of the function H, which is needed to determine the long distance behaviour correlations, cannot be found in Burgers' book. The expressions in (2) as well as the function H (89) apparently cannot be made more explicit, but allow for extracting numerical values and exact asymptotic properties.

The situation considered in this paper is particularly relevant to the non-equilibrium statistical model of ballistic aggregation: it is known (Burgers 1974; Kida 1979) that the dynamics of shocks in Burgers' turbulence is closely related to the dynamics of the aggregating particles. White noise initial distribution of the Burgers velocity field corresponds to Maxwellian initial velocity distribution of the particle undergoing aggregation. Hence, our results also solve this statistical mechanical model. A precise connection between the two problems can only be made in a proper scaling limit since ballistic aggregation always retains the discrete nature of particles whereas the Burgers velocity field describes a continuous medium. This connection is discussed in a companion paper (Frachebourg, Martin & Piasecki 2000) where it is also shown that our solution verifies the hierarchy of kinetic equations that govern the dynamics of the aggregation process.

Comparison with decaying Burgers turbulence arising from other classes of stochastic initial data will be given in the conclusion. The forced Burgers equation under the action of external random forces (see Polyakov 1995; Yakhot & Chechlov 1996; E *et al.* 1997) is not discussed in this paper.

2. General setting

For convenience, we briefly recall the construction of solutions of the Burgers equation in the inviscid limit (see Burgers 1974; Woyczyński 1998 and references therein). Introducing the potential $\partial\Psi(x,t)/\partial x = u(x,t)$ together with the Hopf–Cole transformation

$$\Psi(x,t) = -2\nu \ln \theta(x,t), \tag{4}$$

it is found that the function $\theta(x,t)$ satisfies the linear diffusion equation

$$\frac{\partial}{\partial t}\theta(x,t) = \nu \frac{\partial^2}{\partial x^2}\theta(x,t). \tag{5}$$

It can be readily solved, leading to the explicit solution

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}y \frac{x-y}{t} \exp\left(-\frac{1}{2\nu}F(x,y,t)\right)}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}y \exp\left(-\frac{1}{2\nu}F(x,y,t)\right)} \tag{6}$$

where

$$F(x,y,t) = \frac{(x-y)^2}{2t} - \psi(y), \tag{7}$$

with

$$\psi(y) = -\Psi(y,0) = -\int_0^y \mathrm{d}y'\, u(y',0), \tag{8}$$

which depends upon the initial condition. Burgers turbulence corresponds here to the situation where the initial velocity field $u(x,0)$ is a white-noise process in space ($\langle u(x,0)\, u(y,0)\rangle = (D/2)\delta(x-y)$), or equivalently $\psi(y)$ is a two-sided Brownian motion with diffusion coefficient $D/2$ pinned at $\psi(0) = 0$.

In the inviscid limit $\nu \to 0$, the only contributions of the integrals in equation (6) come from the minima of the function $F(x,y,t)$, which depend on the initial condition through $\psi(y)$,

$$\xi(x,t) = \min_y F(x,y,t), \tag{9}$$

and we obtain

$$u(x,t) = \frac{x - \xi(x,t)}{t}. \tag{10}$$

Owing to the scaling properties of the solution $u(x,t)$, it is trivial to take into account the time dependence of the problem. Indeed, the scaled Brownian motion $t^{\alpha/2}\psi(yt^{-\alpha})$ is equivalent in probability to $\psi(y)$, so that, from (9) and (10) with $\alpha = 2/3$, $t^{2/3}\xi\left(x/t^{2/3}, 1\right)$, is equivalent to $\xi(x,t)$ and $t^{-1/3}u\left(x/t^{2/3}, 1\right)$ is equivalent to $u(x,t)$. We study from now on the fixed time $t = 1$ solution $u(x,1) \equiv u(x)$. It will then always be possible to recover the time-dependent solution through this scaling property as we shall see in the concluding section.

The minimum $\xi(x) \equiv \xi(x,1)$ as a function of x can be found with the help of a nice geometrical interpretation of the solution. Consider a realization of the Brownian motion $\psi(y)$ and a parabola centred at x of equation $(x-y)^2/2 + C$ (see figure 1) and adjust the constant C in order for the parabola to touch $\psi(y)$ without ever crossing it. The coordinate of the contact point is the minimum $\xi(x)$ leading thus to $u(x) = x - \xi(x)$. Then, glide the parabola on the graph of $\psi(y)$ by a continuous change of its centre x and C until it touches it for $x = x_i$ on two contact points ξ_i and ξ_{i+1}. Thus, at $x = x_i$, the function $F(x,y,1)$ has two minima leading to a discontinuity of $u(x)$, called a shock, where $\lim_{\epsilon\to 0} u(x_i - \epsilon) = x_i - \xi_i$ and $\lim_{\epsilon\to 0} u(x_i + \epsilon) = x_i - \xi_{i+1}$. To make $u(x)$ single valued at a shock, we define it to be continuous from the left-hand setting $u(x_i) = x_i - \xi_i$.

A shock is characterized (see figure 1) by its location x_i and two parameters which can be taken as

$$\mu_i = \xi_{i+1} - \xi_i \quad \text{‘strength’}, \qquad \nu_i = x_i - \xi_i \quad \text{‘wavelength’}. \tag{11}$$

(At time t, the strength is usually defined as the discontinuity $\mu_i/t = (\xi_{i+1} - \xi_i)/t$ of $u(x,t)$ at a shock.)

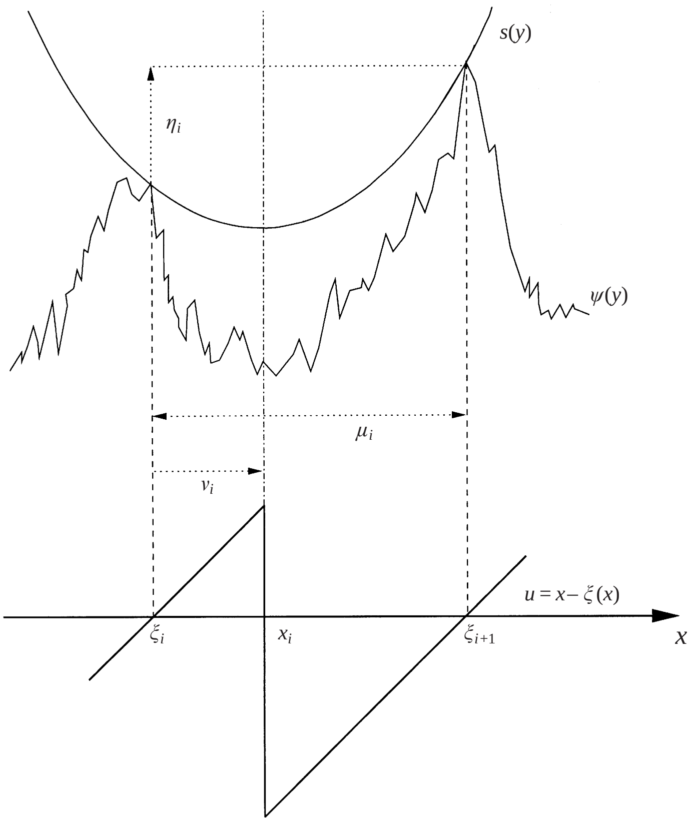

FIGURE 1. Geometrical interpretation of the solution $u(x) = x - \xi(x)$ for a given realization of the Brownian motion $\psi(y)$ which stays below a parabola of equation $s(y) = (y - x_i)^2/2 + C$ but on two contact points $\psi(\xi_i) = (\xi_i - x_i)^2/2 + C$ and $\psi(\xi_{i+1}) = (\xi_{i+1} - x_i)^2/2 + C$. A shock is located at x_i with strength $\mu_i = \xi_{i+1} - \xi_i$ and wavelength $v_i = x_i - \xi_i$ while $\eta_i = \mu_i^2/2 - \mu_i v_i$.

Instead of v_i it will also be convenient to use the parameter η_i

$$\eta_i = \frac{\mu_i^2}{2} - \mu_i v_i, \quad v_i = \frac{\mu_i}{2} - \frac{\eta_i}{\mu_i}. \tag{12}$$

The quantities of interest to be computed are on one hand the joint distribution densities $p_n(x_1, u_1; x_2, u_2; \ldots; x_n, u_n)$ for the Burgers velocity field to have values between u_1 and $u_1 + \mathrm{d}u_1, \ldots, u_n$ and $u_n + \mathrm{d}u_n$ at points $x_1, \ldots, x_n$, when the average is taken over the realizations of the initial condition $u(x, 0)$. On the other hand, we will also consider the joint distribution densities of shocks $\rho_n(x_1, \mu_1, \eta_1; x_2, \mu_2, \eta_2; \ldots; x_n, \mu_n, \eta_n)$. We shall obtain the joint distribution for the Burgers velocity field $u(x)$ from that of the variable $x - \xi(x)$. At time $t = 1$, these two sets of variables coincide and we identify both distributions.

Consider first the one-point distribution density $p_1(x, u)$ where $u(x) = x - \xi(x) = u$. Because of translation invariance, $p_1(x, u) = p_1(0, u) \equiv p_1(u)$ and $u(0) = -\xi(0) = u$. Hence, $p_1(u)$ is the measure of the set of all Brownian paths $\psi(y)$ with $\psi(0) = 0$ that have their first contact (f.c.) (consideration of the first contact (or hitting) point is consistent with the left continuity of $u(x)$. If there is a shock at x_i, $u(x_i) - \lim_{\epsilon \to 0} u(x_i + \epsilon) = \xi_{i+1} - \xi_i > 0$, implying that ξ_i has to be the first contact with the parabola.) with a parabola $y^2/2 + C$ at $\xi(0) = -u$. As the origin of coordinates can be set to be at

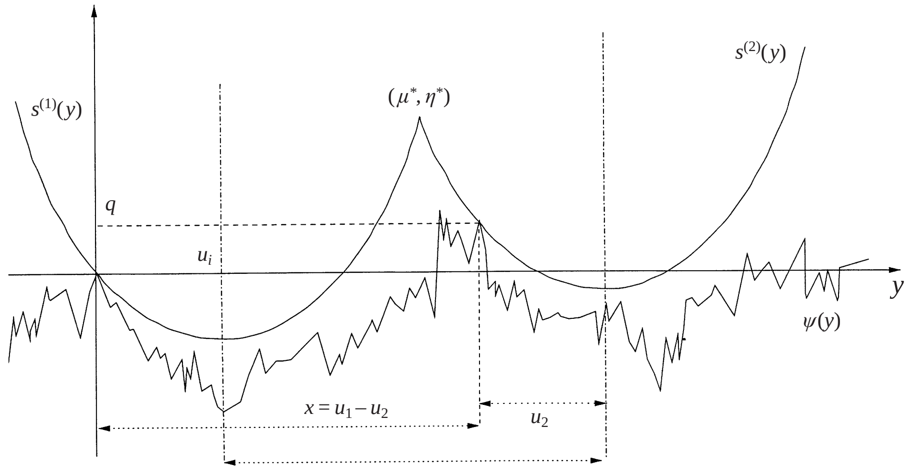

FIGURE 2. Brownian interpretation of the two-point distribution of the velocity field $p_2(x, u_1, u_2)$. The Brownian paths stays under the parabolas $s^{(1)}(y) = y^2/2 - u_1 y$ and $s^{(2)}(y) = (y - x - u_1)^2/2 - u_2^2/2 + q$ but on two contact points $\psi(0) = 0$ and $\psi(x + u_1 - u_2) = q$ where $-(x + u_1 - u_2)(x + u_1 + u_2)/2 \leqslant q \leqslant (x + u_1 - u_2)(x - u_1 - u_2)/2$.

this contact point, it is given by the measure

$$p_1(u) = E\{\psi(y) \leqslant s_u(y), y \in R;\ \text{f.c. with } s_u(y) \text{ at } (0,0)\} \tag{13}$$

of the set of paths that stay below the parabola

$$s_u(y) = \frac{y^2}{2} - uy \tag{14}$$

and have their first contact with it at $\psi(0) = 0$. By first contact in (13), we mean that the path is strictly below the parabola $\psi(y) < s_u(y)$ for $y < 0$, is assigned to pass at $\psi(0) = 0$ and is then such that $\psi(y) \leqslant s_u(y)$ for $y \geqslant 0$. The expectation $E\{\cdots\}$ refers to Brownian paths running in the infinite 'time' interval $-\infty < y < \infty$.

Likewise, the two-point joint density distribution $p_2(0, u_1; x, u_2) \equiv p_2(x, u_1, u_2)$ is the measure of the set of paths with $\psi(0) = 0$ that have a first contact with a parabola $y^2/2 + C_1$ (centred at the origin) at $\xi(0) = -u_1$ and a first contact with a second parabola $(y - x)^2/2 + C_2$ (centred at x) at $\xi(x) = x - u_2$. Once again, we fix the origin at the contact point with the first parabola. Thus, $p_2(x, u_1, u_2)$ is the measure of the set of paths which stay below both the parabolas $s^{(1)}(y) = s_{u_1}(y)$ centred at u_1 and a second parabola $s^{(2)}(y)$ centred at $x + u_1$ of equation $(y - x - u_1)^2/2 + C$, while the paths have a first contact point $\psi(0) = 0$ with $s^{(1)}(y)$ and a first contact point $\psi(x + u_1 - u_2) = q$ with $s^{(2)}(y)$, where $x > 0$, $x + u_1 - u_2 > 0$ (see figure 2). (The case $x + u_1 - u_2 = 0$, i.e. when the two contact points coincide, is discussed in the next section.) In terms of this parameter q the equation of the second parabola is

$$s^{(2)}(y) = \frac{(y - x - u_1)^2}{2} - \frac{u_2^2}{2} + q \tag{15}$$

Now, q is arbitrary except for the constraints that the first contact point with $s^{(1)}(y)$ must be below the second parabola, namely $s^{(2)}(0) \geqslant 0$, and that the first contact point with $s^{(2)}(y)$ must be below the first parabola, namely $s^{(1)}(x + u_1 - u_2) \geqslant q$. This leads to the condition $-q_1 \leqslant q \leqslant q_2$ with

$$\left.\begin{aligned} q_1 &= q_1(x,u_1,u_2) = \tfrac{1}{2}(x+u_1-u_2)(x+u_1+u_2), \\ q_2 &= q_2(x,u_1,u_2) = \tfrac{1}{2}(x+u_1-u_2)(x-u_1-u_2). \end{aligned}\right\} \tag{16}$$

Hence,

$$p_2(x,u_1,u_2) = E\left\{\psi(y) \leqslant s^{(1)}(y),\ \psi(y) \leqslant s^{(2)}(y),\ y \in R;\right.$$
$$\left.\text{f.c. with } s^{(1)}(y) \text{ at } (0,0),\ \text{f.c. with } s^{(2)}(y) \text{ at } (x+u_1-u_2,q),\ -q_1 \leqslant q \leqslant q_2\right\}. \tag{17}$$

The distributions $p_1(u_1)$ and $p_2(x,u_1,u_2)$ have the normalizations

$$\int_{-\infty}^{\infty} \mathrm{d}u_1\, p_1(u_1) = 1 \tag{18}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}u_2\, p_2(x,u_1,u_2) = p_1(u_1), \quad \lim_{x\to 0} p_2(x,u_1,u_2) = \delta(u_1-u_2) \tag{19}$$

The distribution of shocks are defined in the same manner. By translation invariance $\rho_1(x,\mu,\eta) = \rho_1(0,\mu,\eta) \equiv \rho_1(\mu,\eta)$ is independent of x. It is given by the measure of the set of paths that have two contacts with the parabola $s_v(y) = y^2/2 - vy$ (recall that $v = \frac{\mu}{2} - \frac{\eta}{\mu}$), a first contact at $\psi(0) = 0$ and a last contact (l.c.) at $\psi(\mu) = \eta$ (if a path has more than two contacts with the parabola, the shock parameters are obtained in terms of the coordinates of the first and the last contacts) (see figure 1),

$$\rho_1(\mu,\eta) = E\left\{\psi(y) \leqslant s_v(y),\ y \in R;\ \text{f.c. with } s_v(y) \text{ at } (0,0);\ \text{l.c. with } s_v(y) \text{ at } (\mu,\eta)\right\}. \tag{20}$$

The joint distribution $\rho_2(0,\mu_1,\eta_1;x,\mu_2,\eta_2)$ of two shocks at distance x is the set of paths that have two contacts with the parabola $s_{v_1}(y)$ as above and two contacts with another parabola whose characteristics will be given in the next section. Notice that the centres of the two parabolas are separated by a distance x.

All quantities will be eventually expressed in terms of the transition probability kernel for Brownian motion in the presence of parabolic absorbing barriers (Salminen 1988; Groeneboom 1989). Consider the conditional probability density $K_v(\mu_1,\eta_1,\mu_2,\eta_2)$ for the Brownian motion $\psi(y)$, starting from $\psi(\mu_1) = \eta_1$, to end at $\psi(\mu_2) = \eta_2$ while staying under the barrier $\psi(y) < s_v(y) = y^2/2 - vy$ for $\mu_1 \leqslant y \leqslant \mu_2$

$$K_v(\mu_1,\eta_1,\mu_2,\eta_2) = E_{\mu_1,\eta_1}\left\{\psi(y) < s_v(y),\ \mu_1 \leqslant y \leqslant \mu_2;\ \psi(\mu_2) = \eta_2\right\}. \tag{21}$$

It thus satisfies the diffusion equation

$$\partial_{\mu_2} K_v(\mu_1,\eta_1,\mu_2,\eta_2) = \frac{D}{2}\partial^2_{\eta_2} K_v(\mu_1,\eta_1,\mu_2,\eta_2), \tag{22}$$

with $K_v(\mu,\eta_1,\mu,\eta_2) = \delta(\eta_1-\eta_2)$ and $K_v(\mu_1,s_u(\mu_1),\mu_2,\eta_2) = K_v(\mu_1,\eta_1,\mu_2,s_u(\mu_2)) = 0$. To solve this equation, it is convenient to consider the shifted stochastic process $\phi(y) = \psi(y) - s_v(y)$ which is a Brownian motion with a parabolic drift. Clearly,

$$K_v(\mu_1,\eta_1,\mu_2,\eta_2) = \overline{K}(\mu_1,\eta_1 - s_v(\mu_1),\mu_2,\eta_2 - s_v(\mu_2)) \tag{23}$$

where $\overline{K}$ satisfies the diffusion equation with drift

$$\partial_{\mu_2}\overline{K}(\mu_1,\phi_1,\mu_2,\phi_2) = s_v'(\mu_2)\partial_{\phi_2}\overline{K}(\mu_1,\phi_1,\mu_2,\phi_2) + \frac{D}{2}\partial^2_{\phi_2}\overline{K}(\mu_1,\phi_1,\mu_2,\phi_2) \tag{24}$$

with $\overline{K}(\mu,\phi_1,\mu,\phi_2) = \delta(\phi_1-\phi_2)$ and Dirichlet boundary conditions $\overline{K}(\mu_1,0,\mu_2,\phi_2) = \overline{K}(\mu_1,\phi_1,\mu_2,0) = 0$. Equation (24) can be reduced to a diffusion equation with linear potential by the transformation

$$G(\mu_1,\phi_1,\mu_2,\phi_2) = \overline{K}(\mu_1,\phi_1,\mu_2,\phi_2)\exp\left[-\frac{1}{D}\left(\phi_1 s_v'(\mu_1) - \phi_2 s_v'(\mu_2) - \tfrac{1}{2}\int_{\mu_1}^{\mu_2} \mathrm{d}\mu (s_v'(\mu))^2\right)\right]. \tag{25}$$

Then the propagator G is the solution of the equation

$$\left(\frac{\partial}{\partial\mu_2} - \frac{D}{2}\frac{\partial^2}{\partial\phi_2^2} - \frac{1}{D}\phi_2 s_v''(\mu_2)\right) G(\mu_1,\phi_1,\mu_2,\phi_2) = 0 \quad (\phi_1,\ \phi_2 \leqslant 0), \tag{26}$$

with $G(\mu,\phi_1,\mu,\phi_2) = \delta(\phi_1 - \phi_2)$ and Dirichlet boundary conditions at the origin, $G(\mu_1, 0, \mu_2, \phi_2) = G(\mu_1,\phi_1,\mu_2,0) = 0$. Since $s_v''(\mu) = 1$, this equation can be solved with the help of the spectral decomposition of the operator $-\frac{1}{2}D(\partial^2/\partial\phi_2^2) - (1/D)\phi_2$ leading to (Burgers 1974; Salminen 1988; Groeneboom 1989)

$$G(\mu_1,\phi_1,\mu_2,\phi_2) = \left(\frac{2}{D^2}\right)^{1/3} \sum_{k\geqslant 1} \mathrm{e}^{-\omega_k(\mu_2-\mu_1)/(2D)^{1/3}} \frac{\mathrm{Ai}(-(2/D^2)^{1/3}\phi_1 - \omega_k)\mathrm{Ai}(-(2/D^2)^{1/3}\phi_2 - \omega_k)}{(\mathrm{Ai}'(-\omega_k))^2}. \tag{27}$$

The Airy function $\mathrm{Ai}(w)$ (Abramowitz & Stegun 1970), solution of

$$f''(w) - wf(w) = 0, \tag{28}$$

is analytic in the complex w-plane, and has an infinite countable numbers of zeros $-\omega_k$ on the negative real axis, $0 < \omega_1 < \omega_2 < \cdots$.

Finally, coming back to K_v with the help of (23) and (25) and introducing the explicit form (14) of $s_v(y)$ leads to

$$K_v(\mu_1,\eta_1,\mu_2,\eta_2) = G(\mu_1,\phi(\mu_1)),\mu_2,\phi(\mu_2)) \times \exp\left[\frac{1}{D}\left(\phi(\mu_1)(\mu_1 - v) - \phi(\mu_2)(\mu_2 - v) + \frac{(\mu_1 - v)^3}{6} - \frac{(\mu_2 - v)^3}{6}\right)\right] \tag{29}$$

with $\phi(\mu_1) = \eta_1 - s_v(\mu_1)$, $\phi(\mu_2) = \eta_2 - s_v(\mu_2)$. Note the symmetry $K_v(\mu_1,\eta_1,\mu_2,\eta_2) = K_{-v}(-\mu_2,\eta_2,-\mu_1,\eta_1)$.

3. Distributions and transition kernel

In this section we relate the distribution functions to the transition kernel $K_v(\mu_1,\eta_1,\mu_2,\eta_2)$. We first treat the case of a single first contact point by computing the (conditional) probability density

$$E_{\mu_1,\eta_1}\{\psi(y) \leqslant s_v(y), \mu_1 \leqslant y \leqslant \mu_2;\ \text{f.c. with } s_v(y) \text{ at } (\mu, s_v(\mu));\ \psi(\mu_2) = \eta_2\} \tag{30}$$

that a Brownian motion starting at point (μ_1,η_1) ends at (μ_2,η_2) while staying below the parabola $s_v(y)$, and has a first contact point $\psi(\mu) = s_v(\mu)$ at 'time' μ, with $\mu_1 < \mu < \mu_2$ and $\eta_1 < s_v(\mu_1)$ and $\eta_2 < s_v(\mu_2)$. This enables us to write the probability $p_1(u)$ (equation (13)), where the expectation is taken on paths that run in the whole 'time' interval $y \in R$, as

$$p_1(u) = \lim_{\mu_1\to-\infty}\lim_{\mu_2\to\infty} \int_{-\infty}^{s_u(\mu_1)} \mathrm{d}\eta_1 \int_{-\infty}^{s_u(\mu_2)} \mathrm{d}\eta_2 \; E_{\mu_1,\eta_1}\{\psi(y) \leqslant s_u(y), \mu_1 \leqslant y \leqslant \mu_2;\ \text{f.c. with } s_u(y) \text{ at } (0,0);\ \psi(\mu_2) = \eta_2\}. \tag{31}$$

As in the preceding section, it is convenient to consider

$$E_{\mu_1,\phi_1}\{\phi(y)\leqslant 0, \mu_1\leqslant y\leqslant \mu_2,\ \text{f.c. with the origin at}\ \ (\mu,0);\ \phi(\mu_2)=\phi_2\} \\ = -\partial_\mu P_{\mu_1,\phi_1;\mu_2,\phi_2}(\mu) \qquad (32)$$

the quantity corresponding to (30) for the shifted process $\phi(y) = \psi(y) - s_v(y)$. It is the (conditional) probability that a drifted Brownian motion $\phi(\mu)$, starting at point (μ_1, ϕ_1), ends at (μ_2, ϕ_2), stays negative $\phi(y)\leqslant 0$ and has a first contact with the origin at 'time' μ $(\phi(\mu) = 0)$. The desired quantity (30) is obtained by setting $\phi_1 = \phi(\mu_1) = \eta_1 - s_v(\mu_1)$, $\phi_2 = \phi(\mu_2) = \eta_2 - s_v(\mu_2)$ in (32). We have also written that (32) is the density of the probability $P_{\mu_1,\phi_1;\mu_2,\phi_2}(\mu)$ that, under the same constraints, the path has its first contact with the origin at some 'time' larger or equal to μ. This probability is given by (for basic notions on first hitting time see Feller 1971)

$$P_{\mu_1,\phi_1;\mu_2,\phi_2}(\mu) = \int_{-\infty}^{0} \mathrm{d}\phi\, \overline{K}(\mu_1,\phi_1,\mu,\phi)\left[-\partial_\phi \overline{K}(\mu,\phi,\mu_2,\phi_2) - \partial_{\phi_2}\overline{K}(\mu,\phi,\mu_2,\phi_2)\right]. \qquad (33)$$

Indeed one considers the paths starting from (μ_1, ϕ_1) that stay negative up to (μ, ϕ) and then vanish at some 'time' larger or equal to μ. The probability density for the later part is given by the measure of paths staying below the displaced barrier $\phi(y) < \epsilon$ diminished by that of paths staying below the origin $\phi(y) < 0$ as $\epsilon \to 0$, namely by

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(\overline{K}(\mu,\phi+\epsilon,\mu_2,\phi_2+\epsilon) - \overline{K}(\mu,\phi,\mu_2,\phi_2)). \qquad (34)$$

This leads to (33). Introducing (33) into (32) and using the forward diffusion equation (24) as well as its backward equivalent, we find after several integrations by parts that

$$E_{\mu_1,\phi_1}\{\phi(y)\leqslant 0,\ \mu_1\leqslant y\leqslant\mu_2;\ \ \text{f.c. with the origin at}\ (\mu,0);\ \phi(\mu_2)=\phi_2\} \\ = \tfrac{1}{2}D\partial_\phi\overline{K}(\mu_1,\phi_1,\mu,\phi)\partial_\phi\overline{K}(\mu,\phi,\mu_2,\phi_2)\big|_{\phi=0}. \qquad (35)$$

Coming back to the original variables, our probability (30) reads

$$E_{\mu_1,\eta_1}\{\psi(y)\leqslant s_v(y),\ \mu_1\leqslant y\ \leqslant\ \mu_2;\ \ \text{f.c. with}\ s_v(y)\ \text{at}\ (\mu,s_v(\mu));\ \psi(\mu_2)=\eta_2\} \\ = \tfrac{1}{2}D\partial_\eta K_v(\mu_1,\eta_1,\mu,\eta)\partial_\eta K_v(\mu,\eta,\mu_2,\eta_2)\big|_{\eta=s_v(\mu)}, \qquad (36)$$

with K_v given by equation (29).

Let us define the function $J(v)$ to be

$$J(v) = -\sqrt{\tfrac{1}{2}D}\lim_{\mu_2\to\infty}\int_{-\infty}^{s_v(\mu_2)}\mathrm{d}\eta_2\partial_\eta K_v(0,\eta,\mu_2,\eta_2)\bigg|_{\eta=0}. \qquad (37)$$

Then, from (36) and (31) it is straightforward to find the expression of the one-point distribution of the velocity field

$$p_1(u) = J(-u)J(u) \qquad (38)$$

where we used the fact that $K_v(\mu_1,\eta_1,0,\eta) = K_{-v}(0,\eta,-\mu_1,\eta_1)$.

We come now to the two-point function (17) which involves a first contact at $y = 0$ with the parabola $s^{(1)}(y)$ and a first contact at $y = x + u_1 - u_2$ with the second parabola $s^{(2)}(y)$. We consider first the situation where these two contact points are distinct, i.e. when the strict inequality $x + u_1 - u_2 > 0$ holds. Since $u(x)$ has slope equal to one except at the location of shocks, this corresponds to velocity fields with $u(0) = u_1, u(x) = u_2$ that have at least one shock in the interval $[0, x)$.

When $x + u_1 - u_2 > 0$, each contact gives rise to an expression of the form

(36) with appropriate parameters (see figure 2). The first contact with the parabola $s^{(1)}(y) = s_{u_1}(y)$ is as before. After the 'time' μ^* (coordinate of the parabolas intersection $s^{(1)}(\mu^*) = s^{(2)}(\mu^*)$), the paths are found under the second parabola $s^{(2)}(y)$ (15) with corresponding propagator $K_{s^{(2)}}(\mu_1, \eta_1, \mu_2, \eta_2)$ and first contact at $(x + u_1 - u_2, q)$. The corresponding probability is given by the following arrangement

$$\tfrac{1}{2}D\partial_\eta K_{u_1}(\mu_1, \eta_1, 0, \eta)\partial_\eta K_{u_1}(0, \eta, \mu^*, \eta')\big|_{\eta=0} \\ \times \tfrac{1}{2}D\partial_\eta K_{s^{(2)}}(\mu^*, \eta', x + u_1 - u_2, \eta)\partial_\eta K_{s^{(2)}}(x + u_1 - u_2, \eta, \mu_2, \eta_2)\big|_{\eta=q} \quad (39)$$

which has to be integrated on η_1, η_2, η' and q in the appropriate ranges and taken in the limits $\mu_1 \to -\infty$, $\mu_2 \to \infty$. The propagator associated with the second parabola can be written in a coordinate system where the second contact point is again located at the origin, namely

$$K_{s^{(2)}}(\mu_1, \eta_1, \mu_2, \eta_2) = K_{u_2}(\mu_1 - x - u_1 + u_2, \eta_1 - q, \mu_2 - x - u_1 + u_2, \eta_2 - q). \quad (40)$$

Finally, it is found that

$$p_2(u_1, u_2, x) = J(-u_1)H(x, u_1, u_2)J(u_2) \quad (x + u_1 - u_2 > 0), \quad (41)$$

where the function $H(x, v_1, v_2)$ is defined as

$$H(x, v_1, v_2) = \\ \tfrac{1}{2}D \int_{-q_1}^{q_2} \mathrm{d}q \int_{-\infty}^{\eta^*} \mathrm{d}\eta' \, \partial_\eta K_{v_1}(0, \eta, \mu^*, \eta')\bigg|_{\eta=0} \partial_\eta K_{v_2}(\mu^* - x - v_1 + v_2, \eta' - q, 0, \eta)\bigg|_{\eta=0}. \quad (42)$$

The integration limits $q_1 = q_1(x, v_1, v_2)$ and $q_2 = q_2(x, v_1, v_2)$ are given by (16). The intersection point between the two parabolas has coordinate $(\mu^*, \eta^*) = ((q + q_1)/x, \mu^{*2}/2 - v_1\mu^*)$.

We now determine the contribution to $p_2(x, u_1, u_2)$ of the set of velocity fields $u(x)$ that have no shocks in $[0, x)$ (i.e. when $x + u_1 - u_2 = 0$) with the help of the normalization (19). The set of Burgers fields with $u(0) = u_1$ can be divided into the union of two disjoint sets, those having at least one shock in $[0, x)$ and those having no shocks in $[0, x)$. As seen before, the first set corresponds to Brownian paths having two distinct contact points and from the previous discussion its measure is given by $\int_{-\infty}^{u_1+x} \mathrm{d}u_2 J(-u_1)H(x, u_1, u_2)J(u_2)$. The second set corresponds to the case $x+u_1-u_2 = 0$ when Brownian paths have a first contact point $\psi(0) = 0$ at the intersection of the two parabolas $s_{u_1}(y)$ and $s_{u_1+x}(y)$ with measure

$$E\{\psi(y) \leqslant s_{u_1}(y),\ y < 0;\ \psi(y) \leqslant s_{u_1+x}(y),\ y \geqslant 0;\ \text{f.c. with } s_{u_1}(y) \text{ at } (0, 0)\} \\ = J(-u_1)J(u_1 + x). \quad (43)$$

The result (43) is derived by a slight extension of the calculation that led to (38). The measures of these two sets sum up to $p_1(u_1)$

$$J(-u_1)J(u_1 + x) + \int_{-\infty}^{u_1+x} \mathrm{d}u_2 J(-u_1)H(x, u_1, u_2)J(u_2) = p_1(u_1). \quad (44)$$

Hence, we conclude from (19) that the complete form of $p_2(x, u_1, u_2)$ is

$$p_2(x, u_1, u_2) = J(-u_1)\left[\delta(x + u_1 - u_2) + \theta(x + u_1 - u_2)H(x, u_1, u_2)\right] J(u_2). \quad (45)$$

A quantity of interest is the probability density $p_{[0,x)}(u_1)$ for the Burgers field to take the value u_1 at $x = 0$ while there is no shock in the interval $[0, x)$, i.e. $u(x) = u_1 + x$. This is precisely the quantity (43), namely integrating (45) on u_2 with H omitted

$$p_{[0,x)}(u_1) = J(-u_1)J(u_1 + x) \tag{46}$$

and thus

$$p_{[0,x)} = \int_{-\infty}^{\infty} \mathrm{d}u_1 J(-u_1)J(u_1 + x) \tag{47}$$

is the distribution of intervals of length x without shocks.

We turn now to the shocks distribution functions. According to the discussion of the previous section on equation (20), we use equation (36) to write the one-shock distribution function considered as a function of the parameters μ, η (see figure 1)

$$\rho_1(\mu, \eta) = J(-v)I(\mu, \eta)J(-\mu + v) = J\left(-\frac{\mu}{2} + \frac{\eta}{\mu}\right) I(\mu, \eta) J\left(-\frac{\mu}{2} - \frac{\eta}{\mu}\right), \tag{48}$$

where the function $I(\mu, \eta)$ is defined as

$$I(\mu, \eta) = \tfrac{1}{2} D \partial_{\eta_1} \partial_{\eta_2} K_v(0, \eta_1, \mu, \eta_2)\big|_{\eta_1 = 0, \eta_2 = \eta}. \tag{49}$$

The two-shocks distribution $\rho_2(0, \mu_1, \eta_1; x, \mu_2, \eta_2) \equiv \rho_2(x; \mu_1, \eta_1; \mu_2, \eta_2)$ (considered as a function of the shock parameters η_1, μ_1 and η_2, μ_2) can be written as

$$\begin{aligned} \rho_2(x, \mu_1, \eta_1, \mu_2, \eta_2) = {} & J(-v_1)I(\mu_1, \eta_1)\,[\delta(x + v_1 - v_2 - \mu_1) \\ & + \theta(x + v_1 - v_2 - \mu_1)H(x, -\mu_1 + v_1, v_2)]\, I(\mu_2, \eta_2)J(-\mu_2 + v_2), \end{aligned} \tag{50}$$

with $v_i = \mu_i/2 - \eta_i/\mu_i$, $i = 1, 2$, and the functions I, J and H as defined above.

We denote $\rho_2^{(nn)}(x; \mu_1, \eta_1; \mu_2, \eta_2)$ the probability density of two nearest neighbours shocks separated by a distance x; $\rho_2^{(nn)}(x; \mu_1, \eta_1; \mu_2, \eta_2)$ is given by the formula (50) with the H function omitted. Then, the conditional probability density $\rho^{(nn)}(\mu_1, \eta_1 | x, \mu_2, \eta_2)$ that given a shock μ_1, η_1 at $x = 0$, the next shock μ_2, η_2 occurs at $x > 0$ is found to be

$$\begin{aligned} \rho^{(nn)}(\mu_1, \eta_1 | x, \mu_2, \eta_2) &= \frac{\rho_2^{(nn)}(x; \mu_1, \eta_1; \mu_2, \eta_2)}{\rho_1(\mu_1, \eta_1)} \\ &= \delta\left(x - \frac{\eta_1}{\mu_1} + \frac{\eta_2}{\mu_2} - \frac{\mu_1 + \mu_2}{2}\right) \frac{I(\mu_2, \eta_2) J\left(-\dfrac{\mu_2}{2} - \dfrac{\eta_2}{\mu_2}\right)}{J\left(-\dfrac{\mu_1}{2} - \dfrac{\eta_1}{\mu_1}\right)}. \end{aligned} \tag{51}$$

This conditional probability has the normalization

$$\int_0^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}\mu_2 \int_{-\infty}^{\infty} \mathrm{d}\eta_2\, \rho^{(nn)}(\mu_1, \eta_1 | x, \mu_2, \eta_2) = 1 \tag{52}$$

which leads to the following integral relation between the functions I and J

$$J(v) = \int_0^{\infty} \mathrm{d}\mu \int_{-\infty}^{\infty} \mathrm{d}\eta\, \theta\left(\frac{\mu}{2} - \frac{\eta}{\mu} - v\right) I(\mu, \eta) J\left(-\frac{\mu}{2} - \frac{\eta}{\mu}\right). \tag{53}$$

This analysis shows that the one-point and the two-point distribution functions of the Burgers velocity field $u(x)$ as well as of the statistics of shocks are entirely determined by the knowledge of three functions I, J and H defined in (49), (37) and (42). Finally, these last three functions can be computed from the basic transition kernel K_v given by equation (29).

4. The functions I and J and the one-point distribution

In this section we give explicit expressions for the functions $I(\mu,\eta)$ and $J(v)$ defined, respectively, by equations (49) and (37). Through equations (38) and (48), we will then obtain explicit forms for the one-point distribution function of the velocity field $p_1(u)$ and of the shocks $\rho_1(\mu,\eta)$.

Using the form (29) of the transition density K_v in equation (49) we have that

$$I(\mu,\eta) = 2a^3 \exp\left(-a^3\left[\frac{\eta^2}{\mu}+\frac{\mu^3}{12}\right]\right)\mathscr{I}(\mu). \tag{54}$$

We set $a=(2D)^{-1/3}$ and

$$\mathscr{I}(\mu) = \sum_{k\geqslant 1} \mathrm{e}^{-a\omega_k\mu} \tag{55}$$

where $-\omega_k$, $k\geqslant 1$, are the zeros of the Airy function. This last expression has already been found by Burgers (1974). Our point here is to give a closed form for the function $J(v)$ and thus for the one-point distributions. Inserting (29) into (37) and changing the variable $x=-(2/D^2)^{1/3}(\eta_2 - s_v(\mu_2))$ leads to

$$J(v) = \sqrt{a}\lim_{\mu\to\infty}\mathrm{e}^{-a^3[(\mu-v)^3+v^3]/3}\int_0^\infty \mathrm{d}x\,\mathrm{e}^{ax(\mu-v)}\sum_{k\geqslant 1}\mathrm{e}^{-a\omega_k\mu}\frac{\mathrm{Ai}(x-\omega_k)}{\mathrm{Ai}'(-\omega_k)}. \tag{56}$$

It is convenient to introduce the following integral representation of the sum for $\mu>0$

$$\sum_{k\geqslant 1}\mathrm{e}^{-a\omega_k\mu}\frac{\mathrm{Ai}(x-\omega_k)}{\mathrm{Ai}'(-\omega_k)} = \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}}\mathrm{d}w\,\mathrm{e}^{aw\mu}\frac{\mathrm{Ai}(w+x)}{\mathrm{Ai}(w)} = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}w\,\mathrm{e}^{aw\mu}\frac{\mathrm{Ai}(w+x)}{\mathrm{Ai}(w)} \tag{57}$$

where the contour $\mathscr{C}$ runs just above and below the negative real w-axis encircling the zeros of the Airy function. From the asymptotics,

$$\mathrm{Ai}(w) = (4\pi\sqrt{w})^{-1/2}\mathrm{e}^{-2w^{3/2}/3}\left(1+\mathscr{O}(w^{-3/2})\right) \quad (|w|\to\infty,\ |\arg w|<\pi), \tag{58}$$

for $w=|w|\mathrm{e}^{\mathrm{i}\theta}$, $\pi/2\leqslant\theta<\pi$, it can be deduced that $|\mathrm{Ai}(w+x)/\mathrm{Ai}(w)|\sim\exp(-x|w|^{1/2}\cos(\theta/2))$, $|w|\to\infty$, $\cos(\theta/2)>0$. For $\theta=\pi$, the factor $\mathrm{e}^{aw\mu}$ ensures the convergence in (57). Hence, for $\mu>0$, the contour $\mathscr{C}$ can be deformed and it can be shown that the unique contribution to the integral comes from the imaginary axis $-\mathrm{i}\infty<w<\mathrm{i}\infty$ leading to the last part of the identity (57). After exchange of the integrations order, it is found that

$$J(v) = \sqrt{a}\lim_{\mu\to\infty}\mathrm{e}^{-a^3[(\mu-v)^3+v^3]/3}\frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}w\frac{\mathrm{e}^{a\mu w}}{\mathrm{Ai}(w)}\int_0^\infty\mathrm{d}x\,\mathrm{e}^{-ax(v-\mu)}\mathrm{Ai}(x+w). \tag{59}$$

To proceed, we determine first the Laplace transform of $f(x)=\mathrm{Ai}(x+w)$, w fixed,

$$\tilde{f}(s) = \int_0^\infty \mathrm{d}x\,\mathrm{e}^{-xs}f(x). \tag{60}$$

The function $f(x)$ is the solution of the second-order differential equation

$$f''(x)-(x+w)f(x)=0, \tag{61}$$

with $f(0)=\mathrm{Ai}(w)$ and $f'(0)=\mathrm{Ai}'(w)$. The Laplace transform of this equation is

$$\tilde{f}'(s)+(s^2-\omega)\tilde{f}(s) = sf(0)+f'(0), \tag{62}$$

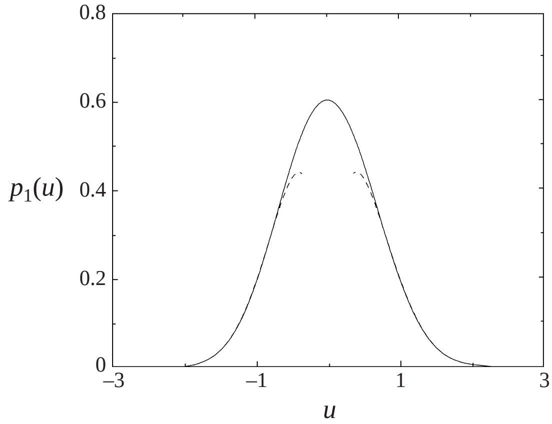

FIGURE 3. The one-point distribution function (68) for the velocity field $p_1(u)$ as a function of u for $D = 1/2$ ($a = 1$). Its asymptotic behaviour (equation (71)), is also plotted (dashed lines).

with solution

$$\tilde{f}(s) = \left(\tilde{f}(0) + \int_0^s \mathrm{d}\sigma\,(\sigma f(0) + f'(0))\mathrm{e}^{-w\sigma+\sigma^3/3}\right)\mathrm{e}^{ws-s^3/3}, \tag{63}$$

$$\tilde{f}(0) = \int_0^\infty \mathrm{d}x\,\mathrm{Ai}(x+w) = -\pi\left[\mathrm{Ai}'(w)\mathrm{Gi}(w) - \mathrm{Ai}(w)\mathrm{Gi}'(w)\right], \tag{64}$$

where $\mathrm{Gi}(w) = \pi^{-1}\int_0^\infty \mathrm{d}t\,\sin(t^3/3 + wt)$ Abramowitz & Stegun (1970).

Inserting this Laplace transform into equation (59) and using various properties of the Airy functions (Abramowitz & Stegun (1970)) leading to the identity

$$\tilde{f}(0) - \int_{-\infty}^0 \mathrm{d}\sigma\,(\sigma f(0) + f'(0))\mathrm{e}^{-\omega\sigma+\sigma^3/3} = 1, \tag{65}$$

we eventually find

$$J(v) = \sqrt{a}\mathrm{e}^{-a^3v^3/3}\mathscr{J}(v) \tag{66}$$

with

$$\mathscr{J}(v) = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}w\frac{\mathrm{e}^{avw}}{\mathrm{Ai}(w)}. \tag{67}$$

Note that this integral is convergent for positive and negative v.

With the explicit form (66) of the function $J(v)$, the one-point distribution function $p_1(u)$ of the velocity field is given by

$$p_1(u) = J(u)J(-u) = a\mathscr{J}(u)\mathscr{J}(-u) \tag{68}$$

which is plotted in figure 3.

Defining the moments of the distribution as $\langle u^n\rangle = \int \mathrm{d}u\,u^n p_1(u)$ we have $\langle u\rangle = 0$ as $p_1(u) = p_1(-u)$ and $\langle u^2\rangle = m_1(D/2)^{2/3}$ with a constant $m_1 \simeq 1.054$. The normalization (18) is verified as, from (68),

$$\int_{-\infty}^{\infty} \mathrm{d}u\,p_1(u) = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}w}{[\mathrm{Ai}(w)]^2} \tag{69}$$

can be shown to be equal to one.

To determine the asymptotic behaviour of $p_1(u)$, we remark that for positive u, we can close the contour in (67) to encircle the poles of the integrand and thus express $\mathscr{J}(u)$ as a sum on the zeros of the Airy function

$$\mathscr{J}(u) = \sum_{k \geqslant 1} \frac{e^{-au\omega_k}}{\mathrm{Ai}'(-\omega_k)} \quad (u > 0). \tag{70}$$

Hence, $\mathscr{J}(u) \sim e^{-au\omega_1}/\mathrm{Ai}'(-\omega_1)$ as $u \to \infty$. The behaviour of $\mathscr{J}(u)$ for $u \to -\infty$ can be determined with the Laplace method to be $\mathscr{J}(u) \sim -2au \exp(a^3u^3/3)$ and so the large $|u|$ behaviour of $p_1(u)$ reads

$$p_1(u) \sim \frac{2a^2|u|}{\mathrm{Ai}'(-\omega_1)} \exp\left(-\frac{a^3|u|^3}{3} - a|u|\omega_1\right) \quad (|u| \to \infty). \tag{71}$$

This result is, of course, compatible with the bounds found in Avellaneda (1995) and Ryan (1998*a*), but cubic bounds cannot be saturated because of the additional exponential decay $\exp(-a|u|\omega_1)$. Starting from a Gaussian distributed initial velocity field $u(x, 0)$, the field immediately evolves to a distribution which is not Gaussian but behaves as equation (71).

Let us turn now to the one-shock distribution function $\rho_1(\mu, \eta)$. Collecting results from equations (48), (54) and (66), we find

$$\rho_1(\mu, \eta) = J\left(\frac{\eta}{\mu} - \frac{\mu}{2}\right) I(\mu, \eta) J\left(-\frac{\eta}{\mu} - \frac{\mu}{2}\right) = 2a^4 \mathscr{J}\left(\frac{\eta}{\mu} - \frac{\mu}{2}\right) \mathscr{I}(\mu) \mathscr{J}\left(-\frac{\eta}{\mu} - \frac{\mu}{2}\right) \tag{72}$$

with $\mathscr{I}$ and $\mathscr{J}$ defined in (55) and (67), respectively.

One can compute the shock strength distribution defined as

$$\rho_1(\mu) = \int_{-\infty}^{\infty} d\eta \, \rho_1(\mu, \eta). \tag{73}$$

Inserting (72) in this last equation, we find after the change of variables $w = i\zeta$ and $\eta' = a\eta/\mu$

$$\rho_1(\mu) = 2a^3 \mu \mathscr{I}(\mu) \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \frac{e^{-ia\mu(\zeta_1+\zeta_2)/2}}{\mathrm{Ai}(i\zeta_1)\mathrm{Ai}(i\zeta_2)} \int_{-\infty}^{\infty} d\eta' e^{i\eta'(\zeta_1 - \zeta_2)} \tag{74}$$

which reduces to

$$\rho_1(\mu) = 2a^3 \mu \mathscr{I}(\mu) \mathscr{H}(\mu) \tag{75}$$

with

$$\mathscr{H}(\mu) = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} dw \frac{e^{-a\mu w}}{\mathrm{Ai}^2(w)}. \tag{76}$$

The form of the shock strength distribution (75) is plotted in figure 4. Notice that $L\langle\mu\rangle$ is the space covered by the shock strength in a box of size L; it is equal to L and thus $\int_0^\infty d\mu \, \mu \rho_1(\mu) = 1$.

We can now determine the behaviour of the shock strength distribution for small and large shocks. For $0 < \mu \ll 1$, we use the normalization condition (18) to find $\mathscr{H}(\mu) = 1 + \mathscr{O}(\mu)$ while the behaviour of $\mathscr{I}(\mu)$ can be determined from the large k asymptotic behaviour of the zeros of the Airy function $\omega_k = (3\pi k/2)^{2/3} + \mathscr{O}(k^{-1/3})$ to give $\mathscr{I}(\mu) \sim (2\sqrt{\pi}(a\mu)^{3/2})^{-1}$. One thus obtains

$$\rho_1(\mu) = \sqrt{\frac{a^3}{\pi\mu}} + O(\mu^{1/2}) \quad (\mu \to 0). \tag{77}$$

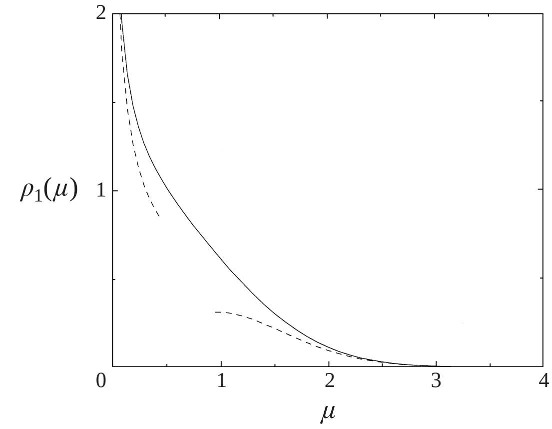

FIGURE 4. Shock strength distribution $\rho_1(\mu)$ for $D = 1/2$ ($a = 1$ in equation (75)). Its asymptotic behaviours (equations (77) and (78)) are also plotted (dashed lines).

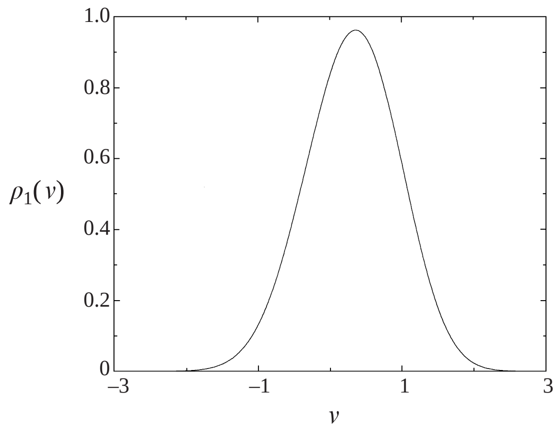

FIGURE 5. Shock wavelength distribution $\rho_1(v)$ for $D = 1/2$ ($a = 1$ in equation (75)).

The divergence $\mu^{-1/2}$, as $\mu \to 0$, has been found in Avellaneda & E (1995) and seen in numerical simulations (Kida 1979).

On the other hand, for large μ, the behaviour of the function $\mathscr{H}(\mu)$ can be estimated by the Laplace method to find $\mathscr{H}(\mu) \sim (\pi a^3 \mu^3)^{1/2} \exp(-a^3\mu^3/12)$. The behaviour of the function $\mathscr{I}(\mu)$ is immediately given by the largest zero of the Airy function to give $\mathscr{I}(\mu) \sim \exp(-\omega_1 a\mu)$. We thus have

$$\rho_1(\mu) = 2\sqrt{\pi} a^{9/2} \mu^{5/2} \exp\left(-\tfrac{1}{12} a^3 \mu^3 - \omega_1 a \mu\right) \quad (\mu \to \infty). \tag{78}$$

Let us consider now the shocks wavelength distribution. The one-shock distribution (72) can be written for the strength-wavelength variables (μ, v) as (the additional μ factor is the Jacobian of the transformation (μ, η) to (μ, v))

$$\rho_1(\mu, v) = 2a^4 \mu \mathscr{J}(-v) \mathscr{I}(\mu) \mathscr{J}(v - \mu). \tag{79}$$

Considering the variable $v' = v - \mu/2$ we find that the (μ, v') distribution is symmetric in v', implying $\langle v' \rangle = 0$ and thus $\langle v \rangle = \langle \mu \rangle/2 = 1/2$. The wavelength distribution $\rho_1(v) = \int_0^\infty d\mu\, \rho_1(\mu, v)$ is plotted in figure 5. Its asymptotic behaviour is found to

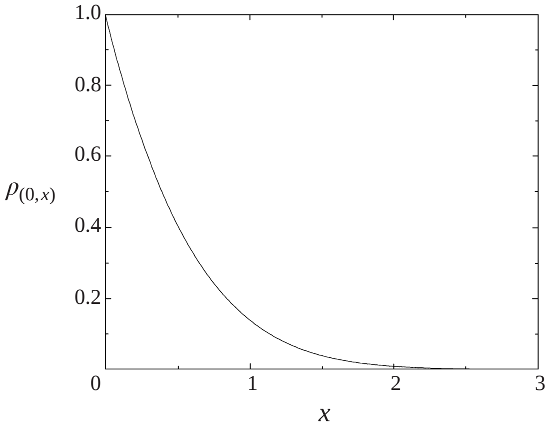

FIGURE 6. Distribution $p_{[0,x)}$ of intervals $[0,x)$ which contains no shocks for $D = 1/2$ ($a = 1$ in equation (80)).

be $\rho_1(v) \sim C_+ v^3 \exp(-a^3v^3/3 - av\omega_1)$, $v \to \infty$, and $\rho_1(v) \sim C_- \exp(a^3v^3/3 + av\omega_1)$, $v \to -\infty$. Note that the wavelength distribution is not symmetrical around $v = 1/2$.

The density distribution $p_{[0,x)}$ of intervals of size x with no shocks (47) is given by

$$
\begin{aligned}
p_{[0,x)} &= \int_{-\infty}^{\infty} \mathrm{d}u_1\, J(-u_1)J(x+u_1) \\
&= \sqrt{\frac{\pi}{ax}} \exp\left(-\frac{a^3x^3}{12}\right) \frac{1}{(2\pi\,\mathrm{i})^2} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\omega_1 \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\omega_2\, \frac{\exp\left(\dfrac{ax}{2}(\omega_1+\omega_2) + \dfrac{(\omega_1-\omega_2)^2}{4ax}\right)}{\mathrm{Ai}(\omega_1)\mathrm{Ai}(\omega_2)}
\end{aligned}
\tag{80}
$$

which is plotted in figure 6. Since $\lim_{x\to 0} p_{[0,x)}(u_1) = p_1(u_1)$, (see equation (46)), and $p_1(u)$ is normalized (18), we have $\lim_{x\to 0} p_{[0,x)} = 1$. Asymptotically, we have for $x \to \infty$

$$
p_{[0,x)} \sim \sqrt{\frac{\pi}{ax}} \frac{\exp\left(-\dfrac{a^3x^3}{12} - a\omega_1 x\right)}{\left[\mathrm{Ai}'(-\omega_1)\right]^2} \left(1 + O\left(\frac{1}{x}\right)\right). \tag{81}
$$

5. Correlations

In this section we study the two-point distributions of the Burgers velocity field and of the shocks in the asymptotic limit $x \to \infty$, keeping all the other arguments fixed. From (45) and (50), we have for x large enough

$$
p_2(x, u_1, u_2) = J(-u_1)H(x, u_1, u_2)J(u_2) \tag{82}
$$

and

$$
\rho_2(x, \mu_1, \eta_1, \mu_2, \eta_2) = J(-v_1)I(\mu_1, \eta_1)H(x, -\mu_1 + v_1, v_2)I(\mu_2, \eta_2)J(-\mu_2 + v_2), \tag{83}
$$

with the functions J and I given by equations (66) and (54) and where the function H is defined by equation (42) with $v_i = \mu_i/2 - \eta_i/\mu_i$.

Our main results are

$$p_2(u_1,u_2,x) - p_1(u_1)p_1(u_2) \sim -\frac{8\sqrt{\pi}}{a^{1/2}x^{5/2}}\exp\left(-\frac{a^3x^3}{12} - a\omega_1 x\right)\exp\left(-a\omega_1(u_1-u_2)\right)\mathscr{J}(-u_1)\mathscr{J}(u_2) \quad (x\to\infty) \tag{84}$$

and similarly for the distribution of shocks

$$\rho_2(\mu_1,\eta_1,\mu_2,\eta_2,x) - \rho_1(\mu_1,\eta_1)\rho_1(\mu_1,\eta_1) \sim -a^{11/2}\frac{32\sqrt{\pi}}{x^{5/2}}\exp\left(-\frac{a^3x^3}{12} - a\omega_1 x\right) \times \exp\left(-a\omega_1(v_1-v_2-\mu_1)\right)\mathscr{J}(-v_1)\mathscr{I}(\mu_1)\mathscr{I}(\mu_2)\mathscr{J}(-\mu_2+v_2) \quad (x\to\infty). \tag{85}$$

We see that long distance correlations are very weak since they are again dominated by the cubic decaying factor $\exp(-a^3x^3/12)$.

Clearly, in view of (82) and (83), this asymptotic behaviour is determined by that of the function $H(x,v_1,v_2)$. First, we write $H(x,v_1,v_2)$ in explicit form by introducing (29) into (42). It is useful to remember that, by the definition of (μ^*,η^*), $\eta^* = s_{v_1}(\mu^*) = q + s_{v_2}(\mu^* - x - v_1 + v_2)$. To bring the expression in the most symmetric form, the change of integration variables

$$\left.\begin{aligned} &\zeta = (D^2/2)^{1/3}(\eta^* - \eta') \quad (0<\zeta<\infty) \\ &r = \frac{1}{\sqrt{x}}\left(q + \frac{v_1^2}{2} - \frac{v_2^2}{2}\right) \quad \left(-\sqrt{x}r_1 \leqslant r \leqslant \sqrt{x}r_2,\ r_1 = \frac{x}{2}+v_1,\ r_2 = \frac{x}{2} - v_2\right)\end{aligned}\right\} \tag{86}$$

turns out to be adequate. Then, with $a = (2D)^{-1/3}$,

$$H(x,v_1,v_2) = 2a^3\exp\left(\frac{-a^3v_1^3 + a^3v_2^3}{3}\right)\sqrt{x}\exp\left(-\frac{a^3x^3}{12}\right)\int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r\exp(-a^3r^2) \times \int_0^\infty \mathrm{d}\zeta\, \mathrm{e}^{a\zeta x}\sum_{k_1,k_2}\exp\left[-a\omega_{k_1}\left(r_1 + \frac{r}{\sqrt{x}}\right) - a\omega_{k_2}\left(r_2 - \frac{r}{\sqrt{x}}\right)\right]\frac{\mathrm{Ai}(\zeta-\omega_{k_1})\mathrm{Ai}(\zeta-\omega_{k_2})}{\mathrm{Ai}'(-\omega_{k_1})\mathrm{Ai}'(-\omega_{k_2})}. \tag{87}$$

Our main concern is to determine the asymptotic behaviour of this expression as $x\to\infty$. We give here the main steps of the calculation while details and justifications are given in the appendices.

To obtain the basic clustering properties of the model, we expect that $\lim_{x\to\infty} H(x,v_1,v_2) = J(v_1)J(-v_2)$ with $J(v)$ given by the integral in the complex plane equation (66). It is therefore natural to replace the sums on the zeros of Airy functions in (87) by appropriate contour integrals, as in § 4,

$$\sum_{k_1,k_2}\exp\left[-a\omega_{k_1}\left(r_1 + \frac{r}{\sqrt{x}}\right) - a\omega_{k_2}\left(r_2 - \frac{r}{\sqrt{x}}\right)\right]\frac{\mathrm{Ai}(\zeta-\omega_{k_1})\mathrm{Ai}(\zeta-\omega_{k_2})}{\mathrm{Ai}'(-\omega_{k_1})\mathrm{Ai}'(-\omega_{k_2})} = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathscr{C}_x}\mathrm{d}w_1\int_{\mathscr{C}_x}\mathrm{d}w_2\exp\left[aw_1\left(r_1+\frac{r}{\sqrt{x}}\right) + aw_2\left(r_2 - \frac{r}{\sqrt{x}}\right)\right]\frac{\mathrm{Ai}(\zeta+w_1)\mathrm{Ai}(\zeta+w_2)}{\mathrm{Ai}(w_1)\mathrm{Ai}(w_2)}. \tag{88}$$

For a given x, the contour $\mathscr{C}_x$ is chosen as the parabola with branches $w^{\pm}(\rho) =$

$-\rho \pm \mathrm{i}\, ax\sqrt{\rho}$, $0 \leqslant \rho < \infty$. This contour will be convenient to determine the large x asymptotics of $H(x, v_1, v_2)$. The integrals (88) on $\mathscr{C}_x$ converge for r fixed because of the exponentially decreasing factors

$$\exp\left[aw_1\left(r_1 + \frac{r}{\sqrt{x}}\right) + aw_2\left(r_2 - \frac{r}{\sqrt{x}}\right)\right],$$

$$(\mathrm{Re}w_1 < 0), \ \left(\mathrm{Re}w_2 < 0,\ r_1 + \frac{r}{\sqrt{x}} > 0,\ r_2 - \frac{r}{\sqrt{x}} > 0\right),$$

(see Appendix A).

Next, we exchange the ζ-integral with the contour integrals to obtain

$$H(x, v_1, v_2) = 2a^3 \exp\left(\frac{-a^3v_1^3 + a^3v_2^3}{3}\right)\sqrt{x}\exp\left(-\frac{a^3x^3}{12}\right)\int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r \exp\left(-a^3r^2\right)$$
$$\times \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathscr{C}_x} \mathrm{d}w_1 \int_{\mathscr{C}_x} \mathrm{d}w_2 \exp\left[aw_1\left(r_1 + \frac{r}{\sqrt{x}}\right) + aw_2\left(r_2 - \frac{r}{\sqrt{x}}\right)\right]\frac{B(ax, w_1, w_2)}{\mathrm{Ai}(w_1)\mathrm{Ai}(w_2)} \quad (89)$$

where

$$B(x, w_1, w_2) = \int_0^{\infty} \mathrm{d}\zeta\, \mathrm{e}^{\zeta x}\mathrm{Ai}(\zeta + w_1)\mathrm{Ai}(\zeta + w_2) \quad (90)$$

is the Laplace transform of a product of Airy functions evaluated at the negative argument $-x$. This Laplace transform is computed in Appendix B and is given as the difference of two terms $B(x, w_1, w_2) = B_1(x, w_1, w_2) - B_2(x, w_1, w_2)$ (see equation (B 9)). We set $H(x) = H_1(x) - H_2(x)$ with $H_1(x)$ (respectively, $H_2(x)$) the contribution to (89) of $B_1(x, w_1, w_2)$ (respectively, $B_2(x, w_1, w_2)$). Then

$$H_1(x) = \frac{a^{5/2}}{\sqrt{\pi}}\exp\left(\frac{-a^3v_1^3 + a^3v_2^3}{3}\right)\int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathscr{C}_x} \mathrm{d}w_1 \int_{\mathscr{C}_x} \mathrm{d}w_2\, h_1(\eta, w_1, w_2) \quad (91)$$

with

$$h_1(r, w_1, w_2) = \frac{\exp\left[-a^3\left(r - \frac{w_1 - w_2}{2a^2\sqrt{x}}\right)^2 + aw_1v_1 - aw_2v_2\right]}{\mathrm{Ai}(w_1)\mathrm{Ai}(w_2)}. \quad (92)$$

It is shown in Appendix C that the multiple integral in (91) is absolutely convergent. As $x \to \infty$, the contour $\mathscr{C}_x$ eventually opens to the imaginary axis of the w-plane. Hence, it can be seen (formally) on (91) that

$$\lim_{x\to\infty} H_1(x) = a\exp\left(\frac{-a^3v_1^3 + a^3v_2^3}{3}\right)\frac{1}{(2\pi\mathrm{i})^2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}w_1 \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}w_2 \frac{\exp(aw_1v_1 - aw_2v_2)}{\mathrm{Ai}(w_1)\mathrm{Ai}(w_2)}$$
$$= J(v_1)J(-v_2), \quad (93)$$

where the function $J(v)$ is defined by equation (66). More precisely, it is found that the asymptotic behaviour of $H_1(x)$ is given by (Appendix C)

$$H_1(x) = J(v_1)J(-v_2) + O\left(\exp\left(-\frac{a^3x^3}{12}(1 + c)\right)\right) \quad (c > 0). \quad (94)$$

Inserting the expression $B_2(x)$ (B 9) in (89) gives

$$H_2(x) = 2a^{5/2} \exp\left(\frac{-a^3 v_1^3 + a^3 v_2^3}{3}\right) \int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r\, \mathrm{e}^{-a^3 r^2}$$
$$\times \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathscr{C}_x} \mathrm{d}w_1 \int_{\mathscr{C}_x} \mathrm{d}w_2 \exp\left(aw_1\left(r_1 + \frac{r}{\sqrt{x}}\right) + aw_2\left(r_2 - \frac{r}{\sqrt{x}}\right)\right) \int_{ax}^{\infty} \mathrm{d}y \sqrt{y}$$
$$\times \exp\left(-\frac{y^3}{12} + \frac{w_1 + w_2}{2}(y - ax) - \frac{(w_1 - w_2)^2}{4}\left(\frac{1}{ax} - \frac{1}{y}\right)\right) \frac{g(y, w_1, w_2)}{\mathrm{Ai}(w_1)\mathrm{Ai}(w_2)}. \quad (95)$$

Because of the convergence factors, $\exp(aw_1(r_1+r/\sqrt{x})+aw_2(r_2-r/\sqrt{x}))$, the contours $\mathscr{C}_x$ can be closed and the corresponding integrals can again be evaluated at the zeros of the Airy functions (the arguments are similar to those given in Appendix A). Then the relation (B 10) allows the result to be simplified to

$$H_2(x) = 2a^{5/2} \exp\left(\frac{-a^3 v_1^3 + a^3 v_2^3}{3}\right) \int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r\, \mathrm{e}^{-a^3 r^2}$$
$$\times \sum_{k_1,k_2} \exp\left(-a\omega_{k_1}\left(r_1 + \frac{r}{\sqrt{x}}\right) - a\omega_{k_2}\left(r_2 - \frac{r}{\sqrt{x}}\right)\right)$$
$$\times \int_{ax}^{\infty} \mathrm{d}y \frac{1}{\sqrt{y}} \exp\left(-\frac{y^3}{12} - \frac{\omega_{k_1} + \omega_{k_2}}{2}(y - ax) - \frac{(\omega_{k_1} - \omega_{k_2})^2}{4}\left(\frac{1}{ax} - \frac{1}{y}\right)\right). \quad (96)$$

To compute the large x behaviour it is convenient to make the change of integration variable $y = z/x^2 + ax$ giving

$$H_2(x) = 2a^2 \exp\left(\frac{-a^3 v_1^3 + a^3 v_2^3}{3}\right) \frac{\exp\left(-\frac{a^3 x^3}{12}\right)}{x^{5/2}} G(x) \quad (97)$$

with

$$G(x) = \int_0^{\infty} \mathrm{d}z \frac{\exp\left(-\frac{z^3}{12x^6} - \frac{az^2}{4x^3} - \frac{a^2 z}{4}\right)}{\sqrt{1 + \frac{z}{ax^3}}} \int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r\, \mathrm{e}^{-a^3 r^2}$$
$$\times \sum_{k_1,k_2} \exp\left(-(\omega_{k_1} - \omega_{k_2})^2 \frac{z}{4a(zx + ax^4)}\right)$$
$$\times \exp\left(-a\omega_{k_1}\left(r_1 + \frac{r}{\sqrt{x}} + \frac{z}{2ax^2}\right) - a\omega_{k_2}\left(r_2 - \frac{r}{\sqrt{x}} + \frac{z}{2ax^2}\right)\right). \quad (98)$$

Letting formally $x \to \infty$ in this formula gives the asymptotic behaviour (details are found in Appendix D)

$$H_2(x) \sim \frac{8\sqrt{\pi}}{a^{3/2} x^{5/2}} \exp\left(\frac{-a^3 v_1^3 + a^3 v_2^3}{3} - a\omega_1(v_1 - v_2)\right) \exp\left(-\frac{a^3 x^3}{12} - a\omega_1 x\right) \quad (99)$$

where $-\omega_1$ is the first zero of the Airy function.

Inserting the asymptotics equations (99) and (94) into the expression for the two-point distributions equations (82) and (83) leads to the results equations (84) and (85).

6. Conclusion

To conclude, we remark that the previous results allow for a complete statistical description of the Burgers field. As mentioned in § 1, for white noise initial data, $u(x)$ is a Markov process as a function of x Avellaneda & E (1995). Thus, with $P(x_2, u_2|x_1, u_1) = p_2(x_2-x_1, u_1, u_2)/p_1(u_1)$ the transition kernel for the Markov process, the n-point distribution can be written as

$$p_n(x_1, u_1; \dots ; x_n, u_n) = P(x_n, u_n|x_{n-1}, u_{n-1}) \dots P(x_2, u_2|x_1, u_1)p_1(x_1, u_1)$$
$$= \frac{\prod_{i=1}^{n-1} p_2(x_{i+1} - x_i, u_i, u_{i+1})}{\prod_{i=2}^{n-1} p_1(u_i)} \quad (n \geqslant 3). \tag{100}$$

On the same line, a complete statistical description of shocks in Burgers solution is obtained through the n-shocks distribution densities which factorize to

$$\rho_n(x_1, \mu_1, \eta_1, \dots , x_n, \mu_n, \eta_n) = \frac{\prod_{i=1}^{n-1} \rho_2(x_{i+1} - x_i, \mu_i, \eta_i; \mu_{i+1}, \eta_{i+1})}{\prod_{i=2}^{n-1} \rho_1(\mu_i, \eta_i)} \quad (n \geqslant 3). \tag{101}$$

The distribution of ordered sequences of next neighbouring shocks is obtained from (101) by omitting the function H in ρ_2, equation (50). Here, factorization follows simply from the Markov property of Brownian motion and the fact that multiple constraints of the form (16) decouple. From the point of view of the hierarchy of kinetic equations that governs the dynamics of shocks, this factorization corresponds to an exact closure of this hierarchy or to an exact propagation of chaos. This is discussed in Frachebourg *et al.* (2000).

As far as the time dependence is concerned, it can be reintroduced via the basic transition kernel (29), which should be computed with $s_v(y)$ replaced by $s_v(y)/t$. Owing to the invariance of the Brownian measure under the change $\psi(y) \to t^{1/3}\psi(y/t^{2/3})$, it is found that $K_v(\mu_1, \eta_1, \mu_2, \eta_2; t) = t^{-1/3}K_{v'}(\mu'_1, \eta'_1, \mu'_2, \eta'_2)$ where the variables are rescaled according to $\mu'_i = \mu_i t^{-2/3}$, $\eta'_i = \eta_i t^{-1/3}$, and $v' = vt^{-2/3}$. From (37), (49) and (42) this implies the transformation laws of the functions J, I and H

$$\left.\begin{aligned} J(v; t) &= t^{-1/3}J(v'), \\ I(\mu, \eta; t) &= t^{-1}I(\mu', \eta'), \\ H(x, v_1, v_2, t) &= t^{-2/3}H(x', v'_1, v'_2) \end{aligned}\right\} \tag{102}$$

where $x' = xt^{-2/3}$. This leads to the time-dependent distributions

$$p_n(x_1, u_1; \dots ; x_n, u_n; t) = t^{n/3}p_n(x'_1, u'_1; \dots ; x'_n, u'_n), \tag{103}$$

with $u'_i = u_i t^{1/3}$, and

$$\rho_n(x_1, \mu_1, \eta_1; \dots ; x_n, \mu_n, \eta_n; t) = t^{-5n/3}\rho_n(x'_1, \mu'_1, \eta'_1; \dots ; x'_n, \mu'_n, \eta'_n). \tag{104}$$

To obtain (103), we recall that the distributions $p_n(x_1, u_1; \dots ; x_n, u_n)$ were calculated from those of the coordinates of the contact points $x_i - \xi_i$. At time $t \neq 1$, $x_i - \xi_i = u_i t$ introduces a Jacobian t^n included in (103) when expressing the distributions as functions of the Burgers field amplitudes u_i. From there, the well-known time-dependent behaviour of some moments of the distributions is recovered, e.g., the energy dissipation per unit of length $\langle u^2(x,t)\rangle \sim t^{-2/3}$, the average number of shocks per unit of length $\sim t^{-2/3}$, the average strength of a shock $\langle \mu/t\rangle \sim t^{-1/3}$.

We mention a few problems that could be approached with the tools developed in this paper. An interesting quantity is the distribution of the velocity differences

$$\mathrm{Prob}\{u(x,t=1)-u(0,t=1)=\Delta u\} = \int \mathrm{d}u p_2(u, u+\Delta u, x) \equiv p(\Delta u, x) \tag{105}$$

and in particular its asymptotics as $\Delta u \to -\infty$, which gives the probability for a large velocity drop over a fixed interval x. This information cannot be extracted in a straightforward manner from the estimations of § 5 since the latter are valid for fixed u and large x and they are not uniform with respect to the field amplitude. The temporal correlations of the field at different times

$$\mathrm{Prob}\{u(x_1,t_1)=u_1, u(x_2,t_2)=u_2\} \equiv p_2(x_1,u_1,t_1,x_2,u_2,t_2) \tag{106}$$

could in principle be analysed by similar methods. They will involve two Brownian propagators $K^{(t_1)}$ and $K^{(t_2)}$ of the form (29) but related to parabolic barriers with different curvatures. An interesting case is white noise initial distribution with a space dependent strength $D = D(x)$ corresponding to inhomogeneous initial data (compactly supported white noise is studied in Tribe & Zabronski). More generally we expect that these methods will work whenever the initial potential $\Psi(x,0)$ is a process with independent increments; all probability distributions will eventually be expressed in terms of the corresponding transition kernel constrained by the parabolic barriers.

The study of the statistics of the Burgers field with other types of initial distributions is the subject of numerous recent works. There is a first natural generalization to the case when the initial potential $\Psi(x,0)$ is a fractional Brownian process

$$\langle(\Psi(x,0)-\Psi(y,0))\rangle = D|x-y|^{\gamma} \quad (0<\gamma<2). \tag{107}$$

For $\gamma = 1$ (the case of the present paper), there is a finite number of shocks in finite intervals (see Avellaneda 1995; Avellaneda & E 1995). The same is expected to hold for all γ, $0<\gamma<2$. In such cases, it has been shown that the distribution of large fields $\mathrm{Prob}\{u(x,t)\geqslant u\}$ has upper and lower bounds of the type $C_1\exp(-C_2u^{4-\gamma}t^{2-\gamma})$ depending on the exponent γ. Similar estimates hold for the distribution of shock strength (Molchan 1997; Ryan 1998*b*). There is a striking difference when it is assumed that the initial Burgers field has itself fractional Brownian statistics (namely $u(x,0)$ obeys (107)). Then, at any time $t>0$, the set points at which shocks occur are expected to be dense; this has been proved for $\gamma = 1$ (She, Aurell & Frisch 1992; Sinai 1992; Bertoin 1998). In this situation there is a dense set of vanishingly small shocks, but the average number of finite size shocks, say $\mu \geqslant \mu_0$, is bounded above and below at fixed time by $C_1\exp(-C_2\mu^{2-\gamma})$ (Molchan 1997). To conclude, we refer to the paper (Gurbatov *et al.* 1997) for a review of the situations where the initial potential $\Psi(x,0)$ is a spatially homogeneous Gaussian process. Kida (1979) initiated this study when the initial potential correlations decrease rapidly at large distance. This is also an instance where in an appropriate scaling limit the distribution and the correlations of the Burgers field can be obtained in closed analytical form (see chap. 5 of Gurbatov *et al.* 1991; Molchanov, Surgailis & Woyczyński 1995).

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Appendix A

We justify in this appendix the equation (88) which replaces the sum on zeros of the Airy function by an integral in the complex plane.

To evaluate $\mathrm{Ai}(w)$ on the branch $w^+(\rho) = -\rho + \mathrm{i}\, ax\sqrt{\rho}$, we start from the formula $\mathrm{Ai}(-w) = \mathrm{e}^{\mathrm{i}\pi/3}\mathrm{Ai}(w\mathrm{e}^{\mathrm{i}\pi/3}) + \mathrm{e}^{-\mathrm{i}\pi/3}\mathrm{Ai}(w\mathrm{e}^{-\mathrm{i}\pi/3})$ (Abramowitz & Stegun 1970) giving (the formula enables us to obtain the asymptotic behaviour of the Airy function $\mathrm{Ai}(w)$ when $\arg w$ approaches π as is the case for $w^{\pm}(\rho)$, $\rho \to \infty$)

$$\begin{aligned} \mathrm{Ai}(w^+(\rho)) &= \mathrm{e}^{\mathrm{i}\pi/3}\mathrm{Ai}(-w^+(\rho)\mathrm{e}^{\mathrm{i}\pi/3}) + \mathrm{e}^{-\mathrm{i}\pi/3}\mathrm{Ai}(-w^+(\rho)\mathrm{e}^{-\mathrm{i}\pi/3}) \\ &\sim \left(\frac{1}{\sqrt{4\pi}(w^+(\rho))^{1/4}}\right)\left(\exp\left(-\mathrm{i}\,\tfrac{2}{3}(-w^+(\rho))^{3/2}\right) + \exp\left(\mathrm{i}\,\tfrac{2}{3}(-w^+(\rho))^{3/2}\right)\right) \end{aligned} \tag{A 1}$$

where we have used the asymptotic behaviour (58) of the Airy function $\mathrm{Ai}(w)$ for $|w| \to \infty$, $\arg w \neq \pi$. As $\rho \to \infty$,

$$(-w^+(\rho))^{3/2} = (\rho - \mathrm{i}\, ax\sqrt{\rho})^{3/2} = \rho^{3/2} - \mathrm{i}\,\tfrac{3}{2}ax\rho - \tfrac{3}{8}a^2x^2\sqrt{\rho} - \mathrm{i}\, a^3\frac{x^3}{16} + \mathrm{O}\left(\frac{x^4}{\sqrt{\rho}}\right) \tag{A 2}$$

Upon inserting (A 2) into (A 1), it can be seen that

$$\left|\frac{1}{\mathrm{Ai}(w^+(\rho))}\right| \leqslant C_{\rho,x}\exp\left(-ax\rho - \frac{a^3x^3}{24}\right) \tag{A 3}$$

with $C_{\rho,x}$ growing at most algebraically with ρ and x. Using $\mathrm{Ai}(w^*) = \mathrm{Ai}^*(w)$ the same estimate on the branch $w^-(\rho)$ is obtained. By a similar calculation, for fixed ζ, $\mathrm{Ai}(\zeta + w^{\pm}(\rho))/\mathrm{Ai}(w^{\pm}(\rho))$ remains bounded as $\rho \to \infty$.

Consider now the finite parabolic contour closed by a circular arc $\mathscr{R}\mathrm{e}^{\mathrm{i}\theta}$ with θ close to π. On this circular arc for large radius $\mathscr{R}$

$$\begin{aligned} \frac{\mathrm{Ai}(\zeta + \mathscr{R}\mathrm{e}^{\mathrm{i}\theta})}{\mathrm{Ai}(\mathscr{R}\mathrm{e}^{\mathrm{i}\theta})} &\sim \left(\frac{\mathscr{R}\mathrm{e}^{\mathrm{i}\theta}}{\zeta + \mathscr{R}\mathrm{e}^{\mathrm{i}\theta}}\right)^{1/4}\exp\left(-\tfrac{2}{3}(\zeta + \mathscr{R}\mathrm{e}^{\mathrm{i}\theta})^{3/2} + \tfrac{2}{3}(\mathscr{R}\mathrm{e}^{\mathrm{i}\theta})^{3/2}\right) \\ &\sim \left(\frac{\mathscr{R}\mathrm{e}^{\mathrm{i}\theta}}{\zeta + \mathscr{R}\mathrm{e}^{\mathrm{i}\theta}}\right)^{1/4}\exp(-\zeta\sqrt{\mathscr{R}}\mathrm{e}^{\mathrm{i}\theta/2}) = \mathscr{O}(1) \end{aligned} \tag{A 4}$$

as $\mathscr{R} \to \infty$ and $\pi/2 \leqslant \theta \leqslant \pi$. Since $(r_1 + r/\sqrt{x}) > 0$, $(r_2 - r/\sqrt{x}) > 0$, the factors $\exp(aw_1(r_1 + r/\sqrt{x}))$ and $\exp(aw_2(r_2 - r/\sqrt{x}))$ decay exponentially fast when w_1 and w_2 are on the contour $\mathscr{C}_x$ or on the circular arc. It is concluded that the integrals on the circular arcs vanish as $\mathscr{R} \to \infty$ so that the sums in (88) can indeed be replaced by the contour integrals.

Appendix B

The integral $B(x)$ (90)

$$B(x) = \int_0^{\infty} \mathrm{d}\zeta\, \mathrm{e}^{\zeta x}\mathrm{Ai}(\zeta + w_1)\mathrm{Ai}(\zeta + w_2) \tag{B 1}$$

is the Laplace transform for a negative argument $-x$ of the product $f(\zeta) = \mathrm{Ai}(\zeta + w_1)\mathrm{Ai}(\zeta + w_2)$ of two Airy functions (omitting w_1 and w_2 from the notation). First, the asymptotic behaviour of $B(x)$ is determined by the Laplace method

$$B(x) \sim \frac{1}{2\sqrt{\pi}} e^{\Phi(x)} \quad (x \to \infty) \tag{B 2}$$

where

$$\Phi(x) = \frac{x^3}{12} - \frac{x}{2}(w_1 + w_2) - \tfrac{1}{2} \ln x - \frac{(w_1 - w_2)^2}{4x}. \tag{B 3}$$

From the property of the Airy function (28), $f(\zeta)$ verifies the fourth-order differential equation

$$f''''(\zeta) - (4\zeta + 2w_1 + 2w_2) f''(\zeta) - 6f'(\zeta) + (w_1 - w_2)^2 f(\zeta) = 0. \tag{B 4}$$

From (B 4), it is found that its Laplace transform for negative arguments satisfies

$$B'(x) - h(x)B(x) = g(x) \tag{B 5}$$

where we remark that

$$h(x) = \Phi'(x) \tag{B 6}$$

and with

$$g(x) = \frac{x}{4} f(0) - \tfrac{1}{4} f'(0) + \frac{1}{4x}\left[(f''(0) - 2(w_1 + w_2)f(0)\right] - \frac{1}{4x^2}\left[f'''(0) - 2f(0) - 2(w_1 + w_2)f'(0)\right]. \tag{B 7}$$

Equation (B 5) can be solved, using also the value (B 2) for $x \to \infty$,

$$B(x) = B_1(x) - B_2(x) \tag{B 8}$$

with

$$B_1(x) = \frac{1}{2\sqrt{\pi}} e^{\Phi(x)}, \quad B_2(x) = e^{\Phi(x)} \int_x^\infty dy e^{-\Phi(y)} g(y). \tag{B 9}$$

Notice that when evaluated at the zeros of the Airy functions $w_1 = -\omega_{k_1}$, $w_2 = -\omega_{k_2}$, $g(y)$ reduces to

$$g(y)\bigg|_{w_1=-\omega_{k_1},\, w_2=-\omega_{k_2}} = \frac{\mathrm{Ai}'(-\omega_{k_1})\mathrm{Ai}'(-\omega_{k_2})}{2y}. \tag{B 10}$$

Appendix C

We consider the multiple integral $H_1(x)$ (91) and show first that it is absolutely convergent. On the contour $w^{\pm}(\rho) = -\rho \pm \mathrm{i}\, ax\sqrt{\rho}$, $0 \leqslant \rho < \infty$, we have

$$\Re\left(r - \frac{w_1 - w_2}{2a^2\sqrt{x}}\right)^2 = \left(r + \frac{\rho_1 - \rho_2}{2a^2\sqrt{x}}\right)^2 - \frac{x}{4a^2}(\sqrt{\rho_1} \pm \sqrt{\rho_2})^2. \tag{C 1}$$

Hence, using $(\sqrt{\rho_1} \pm \sqrt{\rho_2})^2 \leqslant 2(\rho_1 + \rho_2)$ and (A 3), the integrand (92) is bounded by

$$|h_1(r, w_1, w_2)| \leqslant C_{\rho_1,\rho_2,x} \exp\left(-\frac{a^3x^3}{12}\right) \times \exp\left\{-ax(\rho_1 + \rho_2) - a\rho_1 v_1 + a\rho_2 v_2 - a^3\left(r + \frac{\rho_1 - \rho_2}{2a^2\sqrt{x}}\right)^2 + \frac{ax}{4}\left(\sqrt{\rho_1} \pm \sqrt{\rho_2}\right)^2\right\}$$
$$\leqslant C_{\rho_1,\rho_2,x} \exp\left(-\frac{a^3x^3}{12}\right) \exp\left\{-a\rho_1 r_1 - a\rho_2 r_2 - a^3\left(r + \frac{\rho_1 - \rho_2}{2a^2\sqrt{x}}\right)^2\right\} \tag{C 2}$$

with $C_{\rho_1,\rho_2,x}$ increasing at most algebraically, showing that the integral (91) converges absolutely.

To obtain the asymptotic behaviour (94) of $H_1(x)$ we write the integration of $h_1(r, w_1, w_2)$ over r as

$$\int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r\, h_1(r, w_1, w_2) = \left(\int_{-\infty}^{\infty} \mathrm{d}r - \int_{-\infty}^{-\sqrt{x}r_1} \mathrm{d}r - \int_{\sqrt{x}r_2}^{\infty} \mathrm{d}r \right) h_1(r, w_1, w_2). \tag{C 3}$$

The first integration is readily performed to give $J(v_1)J(-v_2)$ (see equation (93)) as $J(v)$ (66) can be represented as an integral on any contour that encircles the zeros of the Airy function, in particular on $\mathscr{C}_x$. Thus, it follows from (91) that

$$H_1(x) - J(v_1)J(-v_2) = -\frac{a^{5/2}}{\sqrt{\pi}} \exp\left(\frac{-a^3 v_1^3 + a^3 v_2^3}{3} \right)$$
$$\times \left(\int_{-\infty}^{-\sqrt{x}r_1} \mathrm{d}r + \int_{\sqrt{x}r_2}^{\infty} \mathrm{d}r \right) \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathscr{C}_x} \mathrm{d}w_1 \int_{\mathscr{C}_x} \mathrm{d}w_2\, h_1(\eta, w_1, w_2). \tag{C 4}$$

Consider the contribution to (C 4) where $r \geqslant \sqrt{x}r_2$ and the branches of $\mathscr{C}_x$ are $w^+(\rho_1), w^+(\rho_2)$. With (C 2) this contribution is majorized by

$$\left| \int_{\sqrt{x}r_2}^{\infty} \mathrm{d}r \int_{w^+} \mathrm{d}w_1 \int_{w^+} \mathrm{d}w_2\, h(r, w_1, w_2) \right| \leqslant \exp\left(-\frac{a^3 x^3}{12} \right) \int_{\sqrt{x}r_2}^{\infty} \mathrm{d}r \int_0^{\infty} \mathrm{d}\rho_1 \int_0^{\infty} \mathrm{d}\rho_2$$
$$\times \left| \frac{\mathrm{d}w^+(\rho_1)}{\mathrm{d}\rho_1} \right| \left| \frac{\mathrm{d}w^+(\rho_2)}{\mathrm{d}\rho_2} \right| C_{\rho_1,\rho_2,x} \exp\left\{ -a\rho_1 r_1 - a\rho_2 r_2 - a^3 \left(r + \frac{\rho_1 - \rho_2}{2a^2\sqrt{x}} \right)^2 \right\} \tag{C 5}$$

We split the ρ_2 integral into the domains $0 \leqslant \rho_2 \leqslant a^2\sqrt{x}r$ and $a^2\sqrt{x}r \leqslant \rho_2 < \infty$. When $0 \leqslant \rho_2 \leqslant a^2\sqrt{x}r$, $\rho_1 \geqslant 0$, $r \geqslant \sqrt{x}r_2 = \sqrt{x}(x/2 - v_2)$ then

$$\left(r + \frac{\rho_1 - \rho_2}{2a^2\sqrt{x}} \right)^2 \geqslant \left(\frac{r}{2} + \frac{\rho_1}{2a^2\sqrt{x}} \right)^2 \geqslant \left(\frac{r}{2} \right)^2 \geqslant \frac{r^2}{8} + \frac{r_2^2 x}{8} \geqslant \frac{r^2}{8} + cx^3 \tag{C 6}$$

where the last inequality holds for x large enough with $c > 0$, and thus

$$\exp\left\{ -a^3 \left(r + \frac{\rho_1 - \rho_2}{2a^2\sqrt{x}} \right)^2 \right\} \leqslant \exp\left\{ -a^3 \frac{r^2}{8} - ca^3 x^3 \right\}. \tag{C 7}$$

On the other hand, when $\rho_2 \geqslant a^2\sqrt{x}r$, $r \geqslant \sqrt{x}r_2$,

$$\rho_2 r_2 \geqslant \tfrac{1}{2}\rho_2 r_2 + \tfrac{1}{2}a^2 r\sqrt{x}r_2 \geqslant \tfrac{1}{2}\rho_2 r_2 + \tfrac{1}{2}a^2 r_2^2 \geqslant \tfrac{1}{2}\rho_2 r_2 + cx^3 \tag{C 8}$$

where the last inequality holds for x large enough with $c > 0$. This leads to

$$\exp(-a\rho_2 r_2) \leqslant \exp\left(-\tfrac{1}{2}a\rho_2 r_2 - acx^3 \right). \tag{C 9}$$

The bounds (C 7) and (C 9) are introduced in (C 5), the remaining r, ρ_1, ρ_2 integrals are convergent and bounded with respect to x (except for a polynomial growth due to $C_{\rho_1,\rho_2,x}$ and the line elements $|\mathrm{d}w(\rho)/\mathrm{d}\rho| = \sqrt{1 + a^2x^2/4\rho}$). The other contributions to (C 4) are treated in the same way. This leads to the result (94).

Appendix D

We determine here the asymptotic behaviour of $G(x)$ (98) for large x. Starting from

$$G(x) = \int_0^\infty \mathrm{d}z \int_{-\sqrt{x}r_1}^{\sqrt{x}r_2} \mathrm{d}r \sum_{k_1 \geqslant 1} \sum_{k_2 \geqslant 1} \mathscr{G}_{k_1,k_2}(z,r;x) \tag{D 1}$$

with

$$\mathscr{G}_{k_1,k_2}(z,r;x) = \frac{\exp\left(-\dfrac{z^3}{12x^6} - \dfrac{az^2}{4x^3} - \dfrac{a^2 z}{4}\right)}{\sqrt{1+\dfrac{z}{ax^3}}} \mathrm{e}^{-a^3r^2} \exp\left(-(\omega_{k_1}-\omega_{k_2})^2 \frac{z}{4a(zx+ax^4)}\right)$$
$$\times \exp\left[-a\omega_{k_1}\left(r_1 + \frac{r}{\sqrt{x}} + \frac{z}{2ax^2}\right) - a\omega_{k_2}\left(r_2 - \frac{r}{\sqrt{x}} + \frac{z}{2ax^2}\right)\right] \tag{D 2}$$

we define

$$F(x) = \mathrm{e}^{a\omega_1 r_1 + a\omega_1 r_2} G(x) \tag{D 3}$$

where $r_1 = x/2 + v_1$, $r_2 = x/2 - v_2$ and $-\omega_1$ is the largest zero of the Airy function. We then decompose $F(x) = F_a(x) + F_b(x) + F_c(x)$ according to the following splitting of the r integration range and the k_1, k_2 summations (for x large):

$$F_a(x) = \int_0^\infty \mathrm{d}z \int_{-r_1}^{r_2} \mathrm{d}r\, \mathrm{e}^{a\omega_1 r_1 + a\omega_1 r_2} \mathscr{G}_{1,1}(z,r;x) \tag{D 4}$$

$$F_b(x) = \int_0^\infty \mathrm{d}z \int_{-r_1}^{r_2} \mathrm{d}r \mathrm{e}^{a\omega_1 r_1 + a\omega_1 r_2} \left(\sum_{k_1 \geqslant 1} \sum_{k_2 \geqslant 1} \mathscr{G}_{k_1,k_2}(z,r;x) - \mathscr{G}_{1,1}(z,r;x)\right) \tag{D 5}$$

$$F_c(x) = \int_0^\infty \mathrm{d}z \left(\int_{r_2}^{\sqrt{x}r_2} \mathrm{d}r + \int_{-\sqrt{x}r_1}^{-r_1} \mathrm{d}r\right) \mathrm{e}^{a\omega_1 r_1 + a\omega_1 r_2} \sum_{k_1 \geqslant 1} \sum_{k_2 \geqslant 1} \mathscr{G}_{k_1,k_2}(z,r;x). \tag{D 6}$$

By dominated convergence, we immediately obtain

$$\lim_{x\to\infty} F_a(x) = \int_0^\infty \mathrm{d}z\, \exp\left(-\frac{a^2 z}{4}\right) \int_{-\infty}^\infty \mathrm{d}r\, \exp(-a^3 r^2) = \frac{4\sqrt{\pi}}{a^{7/2}}. \tag{D 7}$$

We then show below that $F_b(x)$ and $F_c(x)$ vanish as $x \to \infty$ leading to the asymptotic behaviour

$$G(x) \sim \frac{4\sqrt{\pi}}{a^{7/2}} \mathrm{e}^{-a\omega_1(x+v_1-v_2)} \quad (x \to \infty), \tag{D 8}$$

and thus to the behaviour of $H_2(x)$, equation (99).

Since $-r_1 \leqslant r \leqslant r_2$ in the integral (D 5), x can be chosen large enough so that $r_1 + r/\sqrt{x} \geqslant r_1(1-\epsilon)$, $r_2 - r/\sqrt{x} \geqslant r_2(1-\epsilon)$, $\epsilon > 0$. Hence, the k_1, k_2 term of the integrand in (D 5) is less than

$$\mathrm{e}^{a\omega_1 r_1 + a\omega_1 r_2} \mathscr{G}_{k_1,k_2}(z,r;x) \leqslant \mathrm{e}^{-a^3r^2 - a^2 z/4} \mathrm{e}^{-ar_1(\omega_{k_1}(1-\epsilon)-\omega_1)} \mathrm{e}^{-ar_2(\omega_{k_2}(1-\epsilon)-\omega_1)}, \tag{D 9}$$

showing that the joint z, r integrals and k_1, k_2 summations converge. Moreover, since the term $(k_1, k_2) = (1, 1)$ is absent from the integrand in (D 5), there is at least one of the indices strictly greater than one. If both the indices are strictly greater than one, we can conclude that $0 < F_b(x) \leqslant C \exp(-a \min(r_1, r_2)(\omega_2(1-\epsilon) - \omega_1))$ tends to zero exponentially fast as $x \to \infty$ provided that $\epsilon < (\omega_2 - \omega_1)/\omega_2$ with $-\omega_2$ the second zero of the Airy function. If one of the indices is equal to one, say $k_1 = 1$, $k_2 > 1$, we have $0 < F_b(x) \leqslant C\exp(-ar_2(\omega_2(1-\epsilon)-\omega_1)+a\epsilon r_1)$ which tends exponentially to zero as $x \to \infty$ provided that $\epsilon < (\omega_2-\omega_1)/(1+\omega_2)$.

Consider now the integral in (D 6) with $r_2 \leqslant r \leqslant \sqrt{x} r_2$. Since the factor $\exp\left(-(\omega_{k_1}-\omega_{k_2})^2 \ \ (z/4a(zx+ax^4))\right)$ is smaller than one, the k_1, k_2 summations are bounded by a product of $\mathscr{I}$ functions (55). Hence, for $r \geqslant r_2$

$$0 < F_c(x) \leqslant \exp\left(-\frac{a^3 r_2^2}{2}\right)\int_0^\infty \mathrm{d}v \int_0^{\sqrt{x}r_2} \mathrm{d}r \exp\left(-\frac{a^3r^2}{2}\right) \times \mathrm{e}^{a\omega_1 r_1}\mathscr{I}\left(r_1+\frac{r}{\sqrt{x}}+\frac{z}{2ax^2}\right)\mathrm{e}^{a\omega_1 r_2}\mathscr{I}\left(r_2-\frac{r}{\sqrt{x}}+\frac{z}{2ax^2}\right). \quad \text{(D 10)}$$

For $\mathscr{I}(r_1 + r/\sqrt{x} + z/2ax^2)$ we use the bound $\mathscr{I}(r_1 + r/\sqrt{x} + z/2ax^2) \leqslant C\exp(-a\omega_1(r_1 + r/\sqrt{x} + z/2ax^2)) \leqslant C\ \exp(-a\omega_1 r_1)$ since the argument becomes large as $x \to \infty$, whereas for $\mathscr{I}(r_2 - r/\sqrt{x} + z/2ax^2)$ we use the bound (see the discussion leading to equation (77)) $\mathscr{I}\left(r_2 - r/\sqrt{x} + z/2ax^2\right) \leqslant C\left(r_2 - r/\sqrt{x} + z/2ax^2\right)^{-3/2}$ since the argument can become small when r approaches the upper integration limit $\sqrt{x}r_2$. Thus

$$0 < F_c(x) \leqslant C^2\exp\left(-\frac{a^3r_2^2}{2}+a\omega_1 r_2\right)\int_0^{\sqrt{x}r_2}\mathrm{d}r\exp\left(-\frac{a^3r^2}{2}\right)\int_0^\infty\frac{\mathrm{d}z}{\left(r_2-\dfrac{r}{\sqrt{x}}+\dfrac{z}{2ax^2}\right)^{3/2}}$$
$$= C^2 4ax^{5/2}\exp\left(-\frac{a^3r_2^2}{2}+a\omega_1 r_2\right)\int_0^{r_2}\mathrm{d}r'\frac{\exp\left(-\dfrac{a^3}{2}x(r'-r_2)^2\right)}{\sqrt{r'}}. \quad \text{(D 11)}$$

The second line has been obtained by performing the z-integral and changing the integration variable r to $r' = r_2 - r/\sqrt{x}$. This last integral in (D 11) is finite uniformly with respect to x so that with $r_2 = x/2 - v_2$ the bound (D 11) tends to zero in a Gaussian way as $x \to \infty$. These last arguments can be reproduced to show that the integral with $-\sqrt{x}r_1 \leqslant r \leqslant -r_1$ in equation (D 6) tends to zero.

Note added in proof. We thank J. Bertoin for pointing out to us that formulae (67) and (68) for the distribution of the Burgers field can be found in Groeneboom (1989, Corollary 3.3 and 3.4).

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